

ON TWO SPECIFICATIONS OF ASSET  
EQUILIBRIUM IN MACROECONOMIC MODELS

A Note

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### ABSTRACT

Foley's contention that the continuous analogue of the discrete end of period equilibrium specification is ill-formed unless perfect foresight is assumed is shown to depend on a rather awkward specification of the limiting process. A more easily interpretable way of letting the length of the unit period go to zero yields the result that a sufficient condition for the existence of a well-defined continuous analogue is that there be a sequence of temporary equilibria.

## On Two Specifications of Asset Equilibrium in Macroeconomic Models

### A Note\*

In a recent paper<sup>1</sup> Foley has argued that the continuous analogue of the discrete end of period (temporary) equilibrium specification is "ill-formed"<sup>2</sup> unless perfect foresight is assumed. In this note I shall demonstrate that this perfect foresight condition is unnecessarily strong. A sufficient condition for the existence of a well-defined continuous analogue is that there be a sequence of temporary equilibria.

Consider a sequence of unit market periods, labeled  $T = 0, 1, 2, \dots$ . Temporary equilibrium prevails in period 0 when market demand equals market supply, given the economic agents' endowments, technologies, tastes and expectations about the future.<sup>3</sup> In period 1 the expectations held in period 0 about period 1 may either be confirmed or refuted. Incorrect expectations will in all likelihood be revised and temporary equilibrium can be established in period 1, conditional on these revised expectations and on the new values assumed by the other parameters of the single-period equilibrium. This process can be repeated from period to period. A sequence of single-period equilibria does not require that expectations about the future are realized. All that is required is that production, consumption and trading plans can be realized, i.e. that markets clear in each successive period, given the values assumed by the parameters of each unit period. Among these parameters are possibly incorrect expectations about the future.<sup>4</sup>

In what follows I shall use two subscripts to label stock demand functions and expected prices. The first subscript refers to the point in time at which plans are made and expectations are formed. The second subscript refers to the point in time to which plans and expectations refer.

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Notation

$\Delta t$  : the length of the period.

$D^i(T, T + \Delta t)$ : the amount of good  $i$  economic agents plan, at the beginning of the  $T^{\text{th}}$  period, to hold at the end of that period.

$L(T, T + \Delta t)$  : the real stock of money balances economic agents plan, at the beginning of the  $T^{\text{th}}$  period, to hold at the end of that period.

$d^i(T, T)$  : the amount of good  $i$  economic agents plan, at the beginning of the  $T^{\text{th}}$  period, to hold at the beginning of that period.

$S^i(T)$  : the amount of good  $i$  in existence at the beginning of the  $T^{\text{th}}$  period.

$Q^i(T)$  : the amount of good  $i$  economic agents plan to produce during the  $T^{\text{th}}$  period.

$\bar{Q}^i(T)$  : the amount of good  $i$  produced during the  $T^{\text{th}}$  period.

$q^i(t)$  : the instantaneous planned rate of production.

By definition, 
$$Q^i(T) = \int_T^{T+\Delta t} q^i(z) dz.$$

$\bar{q}^i(t)$  : the instantaneous realized rate of production.

By definition, 
$$\bar{Q}^i(T) = \int_T^{T+\Delta t} \bar{q}^i(z) dz.$$

$C^i(T)$  : the amount of good  $i$  economic agents plan to consume during the  $T^{\text{th}}$  period.

$\bar{C}^i(T)$  : the amount of good  $i$  consumed during the  $T^{\text{th}}$  period.

Again we have

$$C^i(T) = \int_T^{T+\Delta t} c^i(z) dz$$

$$\text{and } \bar{C}^i(T) = \int_T^{T+\Delta t} \bar{c}^i(z) dz.$$

$p^e(T, T+r)$  : the general price level of commodities in terms of money expected, at the beginning of the  $T^{\text{th}}$  period, to prevail during the  $(T+r)^{\text{th}}$  period.  $r = 0, 1, 2, \dots$

$p(T)$  : the general price level of commodities during the  $T^{\text{th}}$  period.

Like Foley I assume that  $p^e(T, T) = p(T)$ : the actual and expected current period price level coincide.<sup>5</sup>

$D^i, d^i, S^i, Q^i, \bar{Q}^i, C^i$  and  $\bar{C}^i$  all represent physical quantities. In the case of financial claims this should be interpreted as the number of claims for given amounts of money at specified times (bonds), the number of claims to a share in profits (equity), etc.

Production and consumption should be interpreted broadly so as to include the creation and liquidation of financial claims.

Because we are considering the very general question of the existence of continuous analogues, it is unnecessary to carry along the specific list of arguments that Foley adopted for his asset demand functions.<sup>6</sup>

Foley arrives at his conclusion that the continuous analogue of the discrete end of period equilibrium is ill-defined by introducing beginning of period stock demands into the end of period equilibrium condition. This is both unnecessary and confusing, since the end of period model is being considered and a consistent argument can be made without ever considering beginning of period stock demands.

Consider the end of period equilibrium specification for some good  $i$ :

$$D^i(T, T + \Delta t) + \int_T^{T+\Delta t} \bar{c}^i(z) dz = S^i(T) + \int_T^{T+\Delta t} q^i(z) dz \quad (1)$$

Foley adds and subtracts  $d^i(T, T)$  and divides by  $\Delta t$ :

$$\frac{D^i(T, T + \Delta t) - d^i(T, T)}{\Delta t} + \frac{\int_T^{T+\Delta t} [c^i(z) - q^i(z)] dz}{\Delta t} - \frac{[S^i(T) - d^i(T, T)]}{\Delta t} = 0 \quad (2)$$

Taking the limit as  $\Delta t$  tends to zero he gets on the left-hand side

$$\dot{D}^i(T, T) + c^i(T) - q^i(T) - \lim_{\Delta t \rightarrow 0} \frac{[S^i(T) - d^i(T, T)]}{\Delta t} \quad (3)$$

Foley clearly assumes  $d^i(T, T) \equiv D^i(T, T)$ . Only then is it true that

$$\lim_{\Delta t \rightarrow 0} \left[ \frac{D^i(T, T + \Delta t) - d^i(T, T)}{\Delta t} \right] = \dot{D}^i(T, T). \text{ This creates some serious}$$

conceptual problems. In a discrete period model beginning of period (instantaneous) stock demand and end of period stock demand are qualitatively different concepts.  $d^i(T, T)$  and  $D^i(T, T + \Delta t)$  have clear interpretations.  $D^i(T - \Delta t, T)$ , last period's end of period stock demand is also well-defined, but the meaning of  $D^i(T, T)$  in a discrete period model is unclear. Figure 1 represents Foley's view of portfolio adjustment in the end of period model. At the beginning of the period the portfolioholder attempts to eliminate the instantaneous portfolio disequilibrium  $d^i(T, T) - S^i(T)$ . [ $D^i(T, T)$  is substituted for  $d^i(T, T)$ ]. Then the gap between the beginning of period desired stock and the end of period desired stock  $D^i(T, T + \Delta t) - D^i(T, T)$  is eliminated. The average rate of planned accumulation over the period is

$$\frac{D^i(T, T + \Delta t) - D^i(T, T) + D^i(T, T) - S^i(T)}{\Delta t}. \text{ Foley then notes correctly that}$$

$$\lim_{\Delta t \rightarrow 0} \left( \frac{D^i(T, T) - S^i(T)}{\Delta t} \right) \text{ is not finite unless we can assume that } D^i(T, T) = S^i(T).$$

When there is perfect foresight it is of course legitimate to equate the actual

and desired beginning of period stocks.

There is another way of taking this limit. It has the advantage of not requiring the introduction of beginning of period stock demands [or the ill-defined  $D^i(T, T)$ ] in a discrete end of period model.

Consider now a sequence of single period equilibria. Perfect foresight is not assumed. The nature of the expectations generating mechanism is immaterial. Equations (3) and (4) give the equilibrium conditions for two successive periods:

$$D^i(T, T + \Delta t) + \int_T^{T+\Delta t} [c^i(z) - q^i(z)]dz - S^i(T) = 0 \quad (3)$$

$$D^i(T - \Delta t, T) + \int_{T-\Delta t}^T [c^i(z) - q^i(z)]dz - S^i(T - \Delta t) = 0 \quad (4)$$

Temporary equilibrium in the  $(T - 1)^{th}$  period means that production, consumption and trading plans have been realized during that period, i.e. that  $c^i(z) = \bar{c}^i(z)$  and  $q^i(z) = \bar{q}^i(z)$ . By definition:

$$S^i(T) = S^i(T - \Delta t) + \int_{T-\Delta t}^T [\bar{q}^i(z) - \bar{c}^i(z)]dz \quad (5)$$

This period's stock equals last period's stock plus last period's production

minus last period's consumption. Therefore,  $S^i(T) = S^i(T - \Delta t) + \int_{T-\Delta t}^T [q^i(z) - c^i(z)]dz$ ,

and [by equation (4)]:

$$S^i(T) = D^i(T - \Delta t, T) \quad (6)$$

Rather than going through equation (5), one could jump to equation (6) directly from equation (4). When the  $i^{th}$  market clears during some period, i.e. when trading plans have been realized, the stock of good  $i$  actually

held at the end of that period equals the stock economic agents were planning, at the beginning of the period, to hold at the end of the period.

Adding and subtracting  $D^i(T - \Delta t)$  in (3) and dividing by  $\Delta t$  we get:

$$\frac{D^i(T, T + \Delta t) - D^i(T - \Delta t, T)}{\Delta t} + \int_T^{T+\Delta t} \frac{[c^i(z) - q^i(z)] dz}{\Delta t} - \frac{[S^i(T) - D^i(T - \Delta t, T)]}{\Delta t} = 0 \quad (7)$$

Taking the limit as  $\Delta t \rightarrow 0$ , the continuous analogue of the end of period model, using equation (6), is given by:

$$D^i(T, T) + c^i(T) - q^i(T) = 0 \quad (8)$$

Note that the beginning of period ( instantaneous ) stock demand function was not used in this derivation. Figure 2 represents this two period sequence of temporary equilibria.

The conclusion that a sequence of single-period equilibria is a sufficient condition for the existence of a continuous analogue to the discrete end of period model remains unchanged if the end of period stock demand functions are "real" demand functions in terms of next period's expected prices. I shall illustrate this with the end of period money equilibrium condition. Money will be labeled M.

$$\text{Note that } L(T, T + \Delta t) \equiv \frac{D^M(T, T + \Delta t)}{p^e(T, T + \Delta t)}. \text{ End of period equilibrium}$$

in two successive periods is given by:

$$p^e(T, T + \Delta t) L(T, T + \Delta t) + \int_T^{T+\Delta t} [c^M(z) - q^M(z)] dz - S^M(T) = 0 \quad (9)$$

$$p^e(T - \Delta t, T) L(T - \Delta t, T) + \int_{T-\Delta t}^T [c^M(z) - q^M(z)] dz - S^M(T - \Delta t) = 0 \quad (10)$$



As before, temporary equilibrium in the  $(T - 1)^{\text{th}}$  period means that

$$S^M(T) = S^M(T - \Delta t) + \int_{T-\Delta t}^T [q^M(z) - c^M(z)] dz$$

i.e. (by equation 10)

$$S^M(T) = p^e(T - \Delta t, T) L(T - \Delta t, T) \tag{11}$$

Add and subtract  $p^e(T - \Delta t, T) L(T - \Delta t, T)$  and  $p^e(T, T + \Delta t) L(T - \Delta t, T)$  in equation (9) and use equation (11). This gives:

$$\begin{aligned} p^e(T, T + \Delta t) [L(T, T + \Delta t) - L(T - \Delta t, T)] + p^e(T - \Delta t, T) L(T - \Delta t, T) \\ - S^M(T) + [p^e(T, T + \Delta t) - p^e(T - \Delta t, T)] L(T - \Delta t, T) \\ + \int_T^{T+\Delta t} [c^i(z) - q^i(z)] dz = 0 \end{aligned}$$

Dividing through by  $\Delta t$  and  $p^e(T, T + \Delta t)$  and taking the limit as  $\Delta t \rightarrow 0$  yields the following well-defined continuous flow equilibrium:

$$\dot{L}(T, T) + \frac{\dot{p}^e(T, T)}{p^e(T, T)} \frac{S^M(T)}{p(T)} + \frac{\dot{c}^M(T) - \dot{q}^M(T)}{p(T)} = 0$$

The planned real accumulation of money balances equals the supply of real money balances. This supply comes from new money creation:  $\frac{\dot{q}^M - \dot{c}^M}{p}$  and from capital gains on the existing stock of money:  $-\frac{\dot{p}^e}{p^e} \frac{S^M}{p}$ . 8

### References

- Buiter, W. H. Temporary Equilibrium and Long-Run Equilibrium. Unpublished Ph.D. Thesis, Yale University, 1975.
- Foley, D. K. "On Two Specifications of Asset Equilibrium in Macroeconomic Models." Journal of Political Economy, Vol. 83, No. 2, April, 1975.

### Footnotes

1. D. K. Foley, "On Two Specifications of Asset Equilibrium in Macroeconomic Models," Journal of Political Economy, Vol. 83, No. 2, April 1975, pp. 303-324.
2. See Foley (1975), p. 312.
3. Like Foley, I shall only consider price expectations.
4. It is important to realize that, in the current period, current period prices are assumed to be known with certainty.
5. See Foley (1975), p. 305.
6. When considering the continuous analogue of a specific model, careful attention should of course be devoted to the limiting behavior of the arguments in the behavioral equations.
7. See Foley (1975), pp. 311-312.
8. The way in which we take the limit poses no problems as regards the limiting behaviour of the arguments that Foley specified in the end of period stock demand functions [Foley (1975) p. 307]. In our notation the end of period demand for real money balances is written as:

Footnotes (cont.)

$$L \left[ W^e(T, T + \Delta t), -\frac{p^e(T, T + 2\Delta t) - p^e(T, T + \Delta t)}{\Delta t p^e(T, T + \Delta t)}, \frac{r^e(T, T + \Delta t)}{p_k^e(T, T + \Delta t)} + \frac{p_k^e(T, T + 2\Delta t) - p_k^e(T, T + \Delta t)}{\Delta t p_k^e(T, T + \Delta t)} \right]$$

$W^e$  denotes expected wealth

$r^e$  denotes the expected rentals of capital

$p_k^e$  denotes the expected relative price of capital goods.

We shall consider  $\lim_{\Delta t \rightarrow 0} \frac{L(T, T + \Delta t) - L(T - \Delta t, T)}{\Delta t}$  concentrating on the second

argument. The others follow by analogy.

$$\lim_{\Delta t \rightarrow 0} \frac{L \left[ \dots, -\frac{p^e(T, T+2\Delta t) - p^e(T, T+\Delta t)}{\Delta t p^e(T, T+\Delta t)}, \dots \right] - L \left[ \dots, -\frac{[p^e(T-\Delta t, T+\Delta t) - p^e(T-\Delta t, T)]}{\Delta t p^e(T-\Delta t, T)}, \dots \right]}{\Delta t}$$

We approximate  $p^e(T, T + 2\Delta t)$  by  $p^e(T, T + 2\Delta t) \approx p^e(T, T + \Delta t) + \frac{d}{dt} p^e(T, T + \Delta t)\Delta t$

and  $p^e(T - \Delta t, T + \Delta t)$  by  $p^e(T - \Delta t, T + \Delta t) \approx p^e(T - \Delta t, T) + \frac{d}{dt} p^e(T - \Delta t, T)\Delta t$

[See Foley (1975) p. 313, footnote 3].

$$\begin{aligned} \text{Consider } \lim_{\Delta t \rightarrow 0} \frac{(-) L_2}{\Delta t} & \left[ \frac{p^e(T, T+2\Delta t) - p^e(T, T+\Delta t)}{\Delta t p^e(T, T+\Delta t)} - \frac{[p^e(T-\Delta t, T+\Delta t) - p^e(T-\Delta t, T)]}{\Delta t p^e(T-\Delta t, T)} \right] \\ = \lim_{\Delta t \rightarrow 0} (-) & \frac{L_2}{p^e(T-\Delta t, T)p^e(T, T+\Delta t)(\Delta t)^2} [p^e(T-\Delta t, T)p^e(T, T+2\Delta t) \\ & - p^e(T-\Delta t, T)p^e(T, T+\Delta t) - p^e(T, T+\Delta t)p^e(T-\Delta t, T+\Delta t) \\ & + p^e(T-\Delta t, T)p^e(T, T+\Delta t)] \end{aligned}$$

Footnotes (cont.)

$$\begin{aligned}
 &= \lim_{\Delta t \rightarrow 0} (-) \frac{L_2}{p^e(T-\Delta t, T)p^e(T, T+\Delta t)(\Delta t)^2} \left[ p^e(T-\Delta t, T) \left[ p^e(T, T+\Delta t) + \frac{d}{dt} p^e(T, T+\Delta t)\Delta t \right] \right. \\
 &\quad \left. - p^e(T, T+\Delta t) \left[ p^e(T-\Delta t, T) + \frac{d}{dt} p^e(T-\Delta t, T)\Delta t \right] \right] \\
 &= \lim_{\Delta t \rightarrow 0} (-) \frac{L_2}{p^e(T-\Delta t, T)p^e(T, T+\Delta t)(\Delta t)^2} \left[ p^e(T-\Delta t, T) \frac{d}{dt} p^e(T, T+\Delta t)\Delta t \right. \\
 &\quad \left. - p^e(T, T+\Delta t) \frac{d}{dt} p^e(T-\Delta t, T)\Delta t \right] \\
 &= \lim_{\Delta t \rightarrow 0} (-) L_2 \left( \frac{\left[ \frac{d}{dt} p^e(T, T+\Delta t) / p^e(T, T+\Delta t) \right]}{\Delta t} - \frac{\left[ \frac{d}{dt} p^e(T-\Delta t, T) / p^e(T-\Delta t, T) \right]}{\Delta t} \right) \\
 &= -L_2 \frac{d}{dt} \left( \frac{\frac{d}{dt} p^e(T, T)}{p^e(T, T)} \right)
 \end{aligned}$$

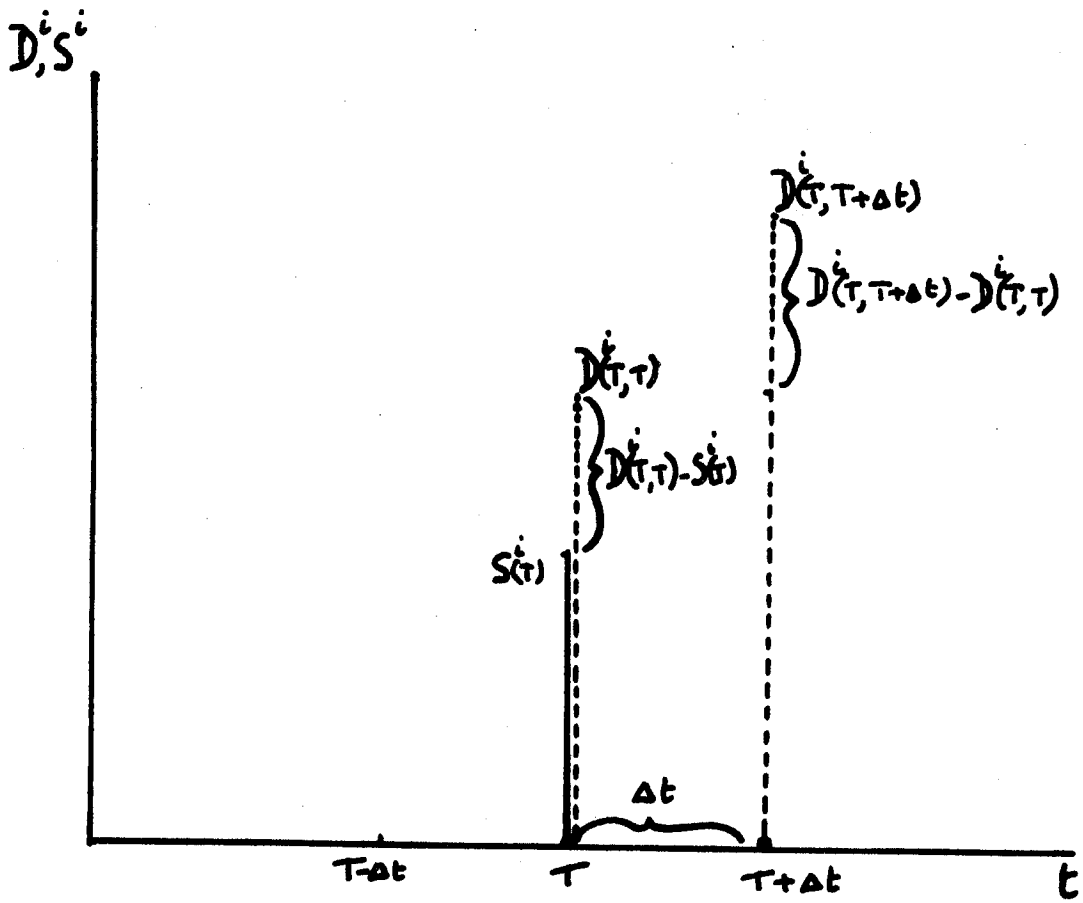


Figure 1

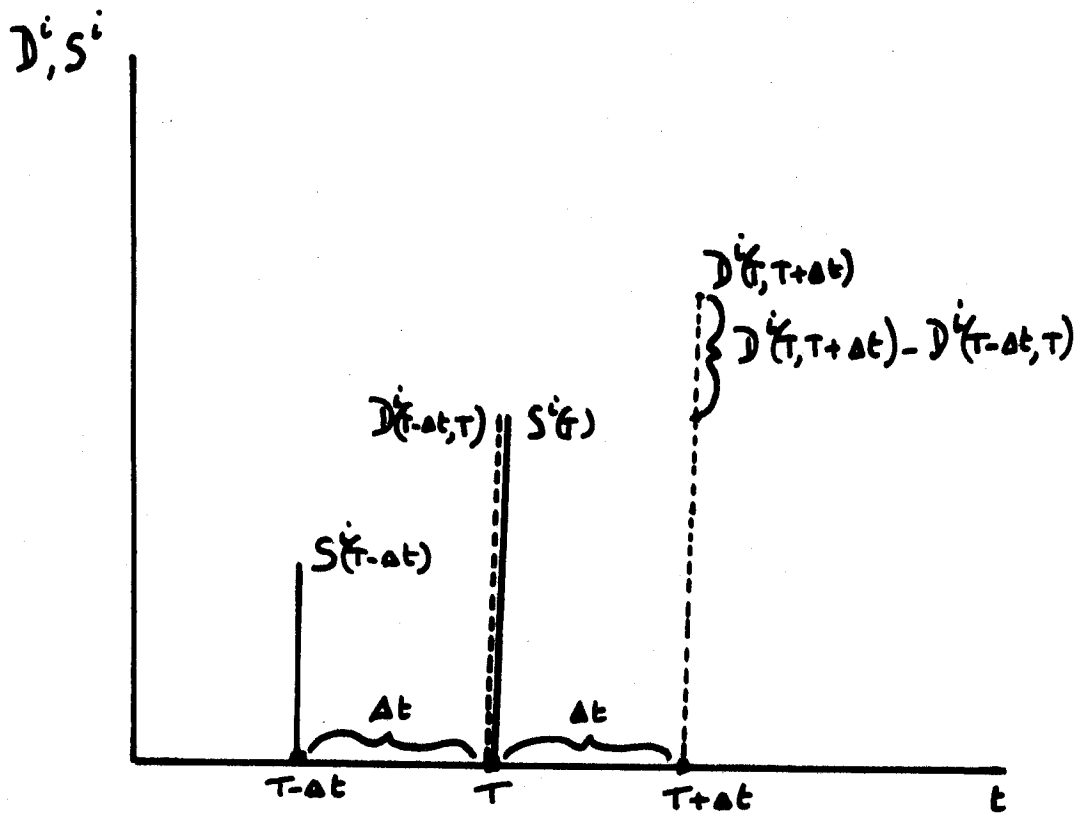


Figure 2