NBER WORKING PAPERS SERIES

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Working Paper No. 3467

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1990

We would like to thank Torsten Persson, Christopher Sims, the participants at the Warwick Summer Research Workshop on "Exchange Rates and Financial Markets", July 9-27, 1990, in particular Renzo Avesani, Giuseppe Bertola, Andy Caplin, Bernard Dumas, Marcus Miller, Rick van der Ploeg, Paul Weller and the participants at the International Finance Workshop at Yale, in particular Giancarlo Corsetti, Vittorio Grilli, Koichi Hamada and Nouriel Roubini. Paolo Pesenti would like to acknowledge financial support from the Ente per gli Studi Monetari, Bancari e Finanziari L. Einaudi, Rome. This paper is part of NBER's research program in International Studies. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

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ABSTRACT

The recent theory of exchange rate dynamics within a target zone holds that exchange rates under a currency band are less responsive to fundamental shocks than exchange rates under a free float, provided that the intervention rules of the Central Bank(s) are common knowledge. These results are derived after having assumed a priori that excess volatility due to rational bubbles does not occur in the foreign exchange market. In this paper we consider instead a setup in which the existence of speculative behavior is a datum the Central Bank has to deal with. We show that the defense of the target zone in the presence of bubbles is viable if the Central Bank accommodates speculative attacks when the latter are consistent with the survival of the target zone itself and expectations are self-fulfilling. These results hold for a large class of exogenous and fundamental-dependent bubble processes. We show that the instantaneous volatility of exchange rates within a band is not necessarily less than the volatility under free float and analyze the implications for interest rate differential dynamics.

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I. Introduction

The recent literature on exchange rate target zones inspired by Krugman's seminal article on the subject (Krugman [1990]) represents a successful marriage of policy relevance and technical innovation. Formal target zones like the exchange rate arrangements of the European Monetary System and informal target zones between the U.S. dollar, the Japanese yen and the D–Mark since the middle of the last decade were a prominent feature of the exchange rate system of the Eighties and promise to be so for the Nineties. The technical modeling innovation consists in the application of the theory of regulated Brownian motion (see e.g. Malliaris and Brock [1982], Harrison [1985], Dixit [1988], Karatzas and Shreve [1988], Dumas [1989] and Flood and Garber [1989]) to the study of the behavior of a floating exchange rate that is constrained by appropriate interventions not to stray outside some given range or target zone.

The literature on this subject is growing very fast. Among the many recent contributions are Krugman [1987], Krugman and Rotemberg [1990], Miller and Weller [1988a,b; 1989; 1990a,b,c], Miller and Sutherland [1990], Froot and Obstfeld [1989a,b], Bertola and Caballero [1989, 1990], Klein [1989], Svensson [1989; 1990a,b], Pesenti [1990], Delgado and Dumas [1990], Lewis [1990], Avesani [1990], Ichikawa, Miller and Sutherland [1990], and Smith and Spencer [1990].

A common element in all of these analyses is that a "fundamentals only" solution is chosen for the exchange rate: The exchange rate, a non–predetermined state variable, is expressed as a function of the current and expected future values of the fundamentals only. The literature in fact concentrates on models with a single fundamental. This fundamental is governed by regulated Brownian motion, that is by unregulated Brownian motion as long as the fundamental stays within the lower and upper intervention thresholds and by (intermittent) interventions that keep the fundamental within these thresholds whenever the unregulated Brownian motion threatens to take the fundamental outside the intervention limits (and thus the exchange rate outside its target zone). This permits the current value of the spot exchange rate to be expressed as a non–linear function of the current value of the fundamental only.

The first step in this argument, the decision to solve for the current value of the exchange rate as a function only of the current and expected future values of the fundamental, means that speculative bubbles are ruled out.

In dynamic linear rational expectations models the arguments for ruling out speculative bubbles are well–known, if not necessarily wholly convincing. In many of the most popular models the bubble processes are non–stationary.[1] These non–stationary bubble processes then lead to non–stationary behavior of state variables such as asset prices. Unbounded growth or decline in these state variables for constant values of the fundamentals is argued to violate, eventually, certain often only implicitly stated feasibility constraints.[2]

Blanchard [1979] has developed an example of a non–stationary (stochastic or deterministic) linear bubble that has a finite expected lifetime. The probability of the bubble surviving for a given period of time declines with the length of that period at a constant exponential rate and approaches zero asymptotically (see also Blanchard and Fischer [1989, chapter 5]). Although the bubble remains non–stationary and its expected value grows exponentially without bound, the finite expected lifetime of this bubble may make it less vulnerable to the standard critique of non–stationary bubbles. Miller and Weller [1990a] and Miller, Weller and Williamson [1989] use the Blanchard bubble to analyze exchange rate behavior (both with and without a target zone) in a stochastic version of the Dornbusch overshooting model (Dornbusch [1976]).

Certain non–linear models also possess non–stationary bubble solutions. For instance, the deflationary and inflationary bubbles studied by Hahn [1982] and by Obstfeld and Rogoff [1983], although obtained in non–linear models, are also non–stationary. In more general non–linear models it it however quite common to find stationary bubbles.[3]

The exchange rate target zone model potentially provides a particularly happy non–linear breeding ground for speculative bubbles. Provided the target zone is credible, that is provided there is certainty that the interventions required to defend the target zone will take place, the exchange rate can neither rise nor fall without bound. There are infinitely many intervention rules that are compatible with the defense of a given target zone. For a rational speculative bubble to exist, the intervention rule must of course be consistent, both within the target zone and at the boundaries, with the behavior of the (regulated) fundamentals. This paper develops a simple and intuitive intervention rule that encompasses as a special case the solution without a bubble, yet is general enough to permit a wide range of exogenous and fundamental–dependent deterministic or stochastic bubbles. With this intervention rule the arguments against bubbles lose their bite. A non–stationary bubble does not now imply non–stationary behavior of the exchange rate. This argument is not new. As early as 1987 William H. Branson, in conversation and discussion, repeatedly raised the possibility that target zones might lead to "indeterminacy" of the exchange rate. This paper can be viewed as an analytical confirmation of his conjecture.

In the next Sections we exposit the simplest version of the credible target zone model and analyze its behavior with or without speculative bubbles.

II. A Target Zone Model With or Without Speculative Bubbles.

(a) The Model.

Our analysis of the place and role of speculative bubbles in an exchange rate target zone requires the exchange rate to be a non–predetermined (forward–looking) state variable, but does not make further demands on descriptive realism. We therefore feel comfortable in using Occam's razor and following the bulk of the literature on target zones by casting the analysis in terms of the simplest possible linear intertemporal substitution equation for the exchange rate and a (continuous time) random walk with drift for the unregulated fundamental. (Notable departures from the cult of extreme simplicity are Miller and Weller [1988a,b; 1989; 1990a,b,c] who analyze stochastic versions of the richer Dornbusch overshooting model.)

The exchange rate process is given in equation (1).

$$s(t)dt = f(t)dt + \alpha^{-1}E_t ds(t) \qquad \alpha > 0. \tag{1}$$

Here $s(t)$ is the natural logarithm of the spot nominal exchange rate and $f(t)$ the fundamental determinant of the exchange rate. E_t is the mathematical expectation operator conditional on the information available at time t to the private sector and the regulator. Both $s(t)$ and $f(t)$ are assumed to be observable at time t, i.e. $E_t s(t) = s(t)$ and $E_t f(t) = f(t)$, and the structure of the model is known.

For the purpose of this paper, the economic interpretation of $f(t)$ is irrelevant except insofar as it affects the credibility of our assumption that, with certainty, the target zone will be successfully defended. A common interpretation of f is in terms of relative nominal money stocks minus relative real–income–related demands for real money balances, i.e.

$$f = m - m^* - k(y - y^*)$$

where m is the logarithm of the home country nominal money stock, y the logarithm of home country real output and k is the (common) income elasticity of money demand. Starred variables denote foreign quantities. In this case α denotes the inverse of the (common) interest semi–elasticity of money demand.[4]

If the two national real outputs are governed by exogenous Brownian motion, intervention means changes in the relative money supplies brought about by monetary financing of public sector financial deficits, through open market operations or by unsterilized foreign exchange market intervention. As shown in Buiter [1989] such intervention policies will in general require adjustments to primary (non–interest) fiscal surpluses to be sustainable indefinitely from a technical point of view (i.e. in order to prevent either exhaustion of foreign exchange reserves or unbounded growth of the public debt). A proof that they are politically feasible and credible is beyond the scope of this paper.

In the absence of intervention, the (unregulated) fundamental $\bar{f}$ would follow a Brownian motion process with drift μ and instantaneous variance σ_f.

$$d\bar{f}(t) = \mu dt + \sigma_f dW_f(t) \qquad \sigma_f > 0.$$

$W_f(t)$ is a standard Wiener process ($dW_f \equiv \lim_{\Delta t \to 0} \sqrt{\Delta t}\, \tilde{n}$ where $\tilde{n}$ is independently and normally distributed with zero mean and unit variance).

We now allow for intervention at the upper and lower boundaries of the exchange rate target zone. The regulated fundamental is governed by

$$df(t) = \mu dt + \sigma_f dW_f(t) - dI_u + dI_\ell \tag{2}$$

I_u and I_ℓ are the cumulative interventions at the upper and lower bounds of the exchange rate target zone, given by s_u and s_ℓ respectively with $-\infty < s_\ell < s_u < \infty$. A detailed specification of these interventions is given below.

(b) The Solution

The general solution of (1) is given in equation (3). $B(t)$ is the value of the speculative bubble at time t.

(3) $$s(t) = \alpha \int_t^{\infty} E_t f(v) e^{-\alpha(v-t)} dv + B(t).$$

The bubble $B(t)$ can be any stochastic or strictly deterministic process satisfying

(4) $$B(t)dt = \alpha^{-1} E_t dB(t).$$

Note that since $s(t)$ is part of the information set at t, so is $B(t)$. We restrict B, where it is stochastic, to follow a diffusion process. Except for the case of the "Blanchard bubble", all examples will in fact restrict it further to be a twice continuously differentiable function of Brownian motions.

Finally, it is important to note that the authorities are assumed to be unable to intervene directly in the bubble process (and indeed in the exchange rate process). They can only regulate the fundamental f.

In the interior of the target zone, the general solution for s as a function of current f and current B is given by[5]

(5) $$s(t) = f(t) + B(t) + \frac{\mu}{\alpha} + A_1(t) e^{\lambda_1(f(t)-C(t))} + A_2(t) e^{\lambda_2(f(t)-C(t))}$$

with

$$\lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 + 2\alpha\sigma_f^2}}{\sigma_f^2}.$$

A_1, A_2 and C are constant as long as s remains in the interior of the target zone, but can change when s is at its upper or lower boundary. In fact we have

$$\text{(6)} \qquad \begin{aligned} A_1(t) &= A_1(\tau) \\ A_2(t) &= A_2(\tau) \\ C(t) &= -B(\tau) \end{aligned}$$

$B(\tau)$ is the value of the bubble at time τ, which is the date of the most recent visit prior to t of the exchange rate s to one of the boundaries, that is

$$\text{(7)} \qquad \tau \equiv \text{Supremum}\{t' \leq t, \text{ such that } s(t') = s_u \text{ or } s(t') = s_\ell\}.$$

We also define s_u^+ to be a value of f both strictly greater than s_u and arbitrarily close to s_u . Similarly, s_ℓ^- is a value of f strictly less than s_ℓ and arbitrarily close to s_ℓ.

We next define two critical values of the bubble, B_u and B_ℓ. B_u is determined, together with A_1, A_2 and f_ℓ, by equations (8a–d).

$$\text{(8a)} \qquad s_u = s_u + \frac{\mu}{\alpha} + B_u + A_1\, e^{\lambda_1(s_u+B_u)} + A_2\, e^{\lambda_2(s_u+B_u)}$$

$$\text{(8b)} \qquad 0 = (1 + \frac{\partial B(\tau)}{\partial f}\Big|_{f=s_u})(1 + \lambda_1 A_1\, e^{\lambda_1(s_u+B_u)} + A_2\, e^{\lambda_2(s_u+B_u)})$$

$$\text{(8c)} \qquad s_\ell = f_\ell + \frac{\mu}{\alpha} + B_u + A_1\, e^{\lambda_1(s_\ell+B_u)} + A_2\, e^{\lambda_2(s_\ell+B_u)}$$

$$\text{(8d)} \qquad 0 = (1 + \frac{\partial B(\tau)}{\partial f}\Big|_{f=f_\ell})(1 + \lambda_1 A_1\, e^{\lambda_1(s_\ell+B_u)} + A_2\, e^{\lambda_2(s_\ell+B_u)}).$$

Analogously, B_ℓ is determined, together with A_1, A_2 and f_u from equations (9a–d).

(9a) $$s_u = f_u + \frac{\mu}{\alpha} + B_\ell + A_1\, e^{\lambda_1(s_u+B_\ell)} + A_2\, e^{\lambda_2(s_u+B_\ell)}$$

(9b) $$0 = \left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{f=f_u}\right)\left(1 + \lambda_1\, A_1\, e^{\lambda_1(s_u+B_\ell)} + A_2\, e^{\lambda_2(s_u+B_\ell)}\right)$$

(9c) $$s_\ell = s_\ell + \frac{\mu}{\alpha} + B_u + A_1\, e^{\lambda_1(s_\ell+B_\ell)} + A_2\, e^{\lambda_2(s_\ell+B_\ell)}$$

(9d) $$0 = \left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{f=s_\ell}\right)\left(1 + \lambda_1 A_1\, e^{\lambda_1(s_\ell+B_\ell)} + A_2\, e^{\lambda_2(s_\ell+B_\ell)}\right).$$

If $B_u > B(\tau) > B_\ell$ then $A_1(\tau)$ and $A_2(\tau)$ are defined, together with $f_u(\tau)$ and $f_\ell(\tau)$ by equations (10a–d).

(10a) $$s_u = f_u(\tau) + B(\tau) + \frac{\mu}{\alpha} + A_1(\tau) e^{\lambda_1[f_u(\tau)+B(\tau)]} + A_2(\tau) e^{\lambda_2[f_u(\tau)+B(\tau)]}$$

(10b) $$s_\ell = f_\ell(\tau) + B(\tau) + \frac{\mu}{\alpha} + A_1(\tau) e^{\lambda_1(f_\ell(\tau)+B(\tau))} + A_2(\tau) e^{\lambda_2(f_\ell(\tau)+B(\tau))}$$

(10c) $$0 = \left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{f=f_u}\right)\left(1 + \lambda_1\, A_1(\tau) e^{\lambda_1(f_u(\tau)+B(\tau))} + \lambda_2 A_2(\tau) e^{\lambda_2(f_u(\tau)+B(\tau))}\right)$$

(10d) $$0 = \left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{f=f_\ell}\right)\left(1 + \lambda_1\, A_1(\tau) e^{\lambda_1(f_\ell(\tau)+B(\tau))} + \lambda_2\, A_2(\tau) e^{\lambda_2(f_\ell(\tau)+B(\tau))}\right)$$

When $B(\tau) \geq B_u$, $A_1(\tau)$ and $A_2(\tau)$ are determined by equations (11a,b).

$$s_u = s_u^+ + \frac{\mu}{\alpha} + B(\tau) + A_1(\tau)\, e^{\lambda_1 s_u^+} + A_2(\tau)\, e^{\lambda_2 s_u^+} \tag{11a}$$

$$0 = \left(1 + \left.\frac{\partial B(\tau)}{\partial f}\right|_{f=s_u^+}\right)\left(1 + \lambda_1 A_1\, e^{\lambda_1 s_u^+} + \lambda_2 A_2\, e^{\lambda_2 s_u^+}\right). \tag{11b}$$

When $B(\tau) \leq B_\ell$, $A_1(\tau)$ and $A_2(\tau)$ are determined by equations (11c,d).

$$s_\ell = s_\ell^- + \frac{\mu}{\alpha} + B(\tau) + A_1(\tau)\, e^{\lambda_1 s_\ell^-} + A_2(\tau)\, e^{\lambda_2 s_\ell^-} \tag{11c}$$

$$0 = \left(1 + \left.\frac{\partial B(\tau)}{\partial f}\right|_{f=s_\ell^-}\right)\left(1 + \lambda_1 A_1\, e^{\lambda_1 s_\ell^-} + \lambda_2 A_2\, e^{\lambda_2 s_\ell^-}\right). \tag{11d}$$

Equations (10a,b) define f_u and f_ℓ for given s_u, s_ℓ and B. Equations (10c,d) are the familiar tangency or no–arbitrage conditions often referred to, inaccurately, as 'smooth pasting' conditions, that expected interventions in f should not change the value of the exchange rate. As we shall see in Section IV below, the value of the bubble may depend on the current value of the unregulated fundamental. The conditions (10c–d), (11b) and (11d) allow for this. Note that for $B_u < B < B_\ell$, A_1 and A_2 are independent of the value of the bubble, and therefore constant over time. It is also readily seen that in this case

$$\frac{d f_u}{dB} = \frac{d f_\ell}{dB} = -1.$$

Equations (11a,b) characterize a one–sided 'smooth pasting' condition at the upper boundary ($s = s_u$) only. Equations (11c,d) characterize a one–sided 'smooth pasting' condition at the lower boundary ($s = s_\ell$) only. These non–standard boundary conditions will be interpreted further below.

We now detail the interventions at the upper and lower boundaries of the exchange rate target zone. From equation (5), the exchange rate in the target zone can be written as $s(t) = g(f(t), B(\tau)) + B(t)$. The bubble at time t can be a function of the current unregulated fundamental $\bar{f}(t)$ and/or of other variables $z(t)$. Note that $z(t)$ could be a function of past or expected future values of the unregulated fundamental. Therefore $B(t) = b(\bar{f}(t), z(t))$. The exchange rate in the target zone can therefore be written as $s = h(f, \cdot)$, where $\frac{\partial h}{\partial f} \equiv \frac{\partial g}{\partial f} + \frac{\partial b}{\partial \bar{f}} \frac{\partial \bar{f}}{\partial f}$. For simplicity we restrict the analysis to those bubbles for which, even if $B(t)$ depends on $\bar{f}(t)$, the function $h(f,\cdot)$, retains its familiar s shape with a unique interior minimum and a unique interior maximum. In the case where $B_u < B(\tau) < B_\ell$, f_u is therefore the value of f for which $s = s_u$ and the h function has an interior maximum, that is f_u is defined by $h(f_u; \cdot) = s_u$, $\frac{\partial h}{\partial f}(f_u;\cdot) = 0$ and $\frac{\partial^2 h}{\partial f^2}(f_u; \cdot) < 0$. Note that because of the bubble, s may reach s_u at values of fundamental different from f_u. These will be denoted f_u^*. Similarly, s may reach s_ℓ at values of f different from f_ℓ. These will be denoted f_ℓ^*.

We consider the simplest possible example of a target zone. A plausible interpretation of the formal model is that there is a fully credible commitment that the value of the exchange rate will be contained within an exogenously given bounded range, that is $-\infty < s_\ell \leq s \leq s_u < \infty$. The interventions in the fundamental process that keep the exchange rate within this range occur only when the exchange rate is at the boundaries. Two distinct kinds of interventions are required to defend the target zone in the presence of a bubble.

The first is a pair of infinitesimal reflecting interventions, $-dI_u < 0$ at the upper boundary if $f = f_u$ and $dI_\ell > 0$ at the lower boundary of the target zone if $f = f_\ell$. These negative infinitesimal interventions at the upper boundary $s = s_u$ and positive infinitesimal interventions at the lower boundary $s = s_\ell$, correspond to the original scheme analyzed by Krugman [1990]. If interventions in the interior of the target zone ("intramarginal" interventions) were permitted, a pair of reflecting interventions of exogenously given finite magnitudes, $-\Delta I_u < 0$ and $\Delta I_\ell > 0$ could be allowed (see Flood and Garber [1989]).

The second class of interventions comprises interventions of finite magnitude, that are functions of the change in the magnitude of the bubble since the previous visit to the boundaries. They too occur only when the exchange rate is at the boundaries of the target zone. They always are in the opposite direction from the infinitesimal (type I) interventions at the same boundary.

We interpret these (type II) discrete interventions as the passive accommodation of rational speculative attacks at the boundaries of the target zone, that are consistent with the survival of the target zone. We refer to such stock–shift portfolio reshuffles as "sustaining" or "friendly" speculative attacks.

The behavior of the regulated fundamental is given in equations (12a) to (12c). The conditions in (12a), (12b), (12c(α)) and (12c(γ)) are the standard ones for regulated Brownian motion (see e.g. Harrison [1985, p. 80]). If $f_u > s_u$ and if $f = f_u$ when $s = s_u$, they characterize a reflecting barrier at $f = f_u$ (given by (12a), (12b) and (12c(α)). If $f_\ell < s_\ell$ and if $f = f_\ell$ when $s = s_\ell$, they characterize a reflecting barrier at $f = f_\ell$ given by (12a), (12b) and (12c(γ)).

The other conditions are non–standard and are required for our analysis of rational bubbles in a target zone. If $f_u > s_u$, they characterize (discrete) accommodated speculative attacks from $f = f_u^*$ to $f = f_u$, followed by (infinitesimal) reflection at $f = f_u$ (given by (12a), (12b) and (12c(β)). If $f_\ell < s_\ell$, they characterize

accommodated speculative attacks from $f = f_\ell^*$ to $f = f_\ell$, followed by **reflection** at $f = f_\ell$ (given by (12a), (12b) and (12c(δ)). If $f_u \leq s_u$, they characterize **accommodated speculative attacks** from $f = f_u^*$ to $f = s_u^+$ (given by (12a), (12b) and (12c(ϵ)). If $f_\ell \geq s_\ell$, they characterize **accommodated speculative attacks from** $f = f_\ell^*$ to $f = s_\ell^-$, (given by (12a), (12b) and 12c(η)). The reasons for these non–standard features will become clear below.

We restrict attention to starting states $s \in [s_\ell, s_u]$ and starting date $t = 0$ and define $t\_$ as the instant immediately before t, that is

$$t\_ \equiv \lim_{\substack{\Delta t \to 0 \\ \Delta t > 0}} (t - \Delta t) \quad .$$

The regulated fundamental is formally defined as follows:

(12a) $$f(t) \equiv \mu t + \int_0^t dW_f - I_u(t) + I_\ell(t) \quad \text{for all } t \geq 0$$

(12b) I^u and I^ℓ are constant if $s_\ell < s < s_u$.

(12c) I^u and I_ℓ vary when $s = s_u$ or $s = s_\ell$:

(I) If $f_u > s_u$ and $f_\ell < s_\ell$ then:

(α) If $s = s_u$ and $f = f_u$ then I^u is continuous and increasing.

(β) If $s(t\_) = s_u$ and $f(t\_) = f_u^*$, then $f(t) = f_u(t)$ and (α) applies henceforth.

(γ) If $s = s_\ell$ and $f = f_\ell$ then I^ℓ is continuous and increasing.

(δ) If $s(t\_) = s_\ell$ and $f(t\_) = f_\ell^*$, then $f(t) = f_\ell(t)$ and (γ) applies henceforth.

(IIa) If $f_u \leq s_u$ then

(ϵ) If $s(t\_) = s_u$ then $f(t) = s_u^+$

If $s = s_\ell$ then (γ) or (δ) apply.

(IIb) If $f_\ell \geq s_\ell$ then

(η) If $s(t\_) = s_\ell$ then $f(t) = s_\ell^-$

If $s = s_u$ then (α) or (β) apply.

It will be apparent that, for any size bubble we can calculate exactly both the magnitudes of the passive or accommodating discrete interventions required to allow the target zone to survive and the values of the fundamentals at which they will occur.

In our rudimentary model, there is no possibility of the speculative bubble influencing the behavior of the fundamental. A bubble in this model can only influence the exchange rate directly. If the process governing the unregulated fundamental included the exchange rate as an argument then, because the bubble affects the exchange rate, it will indirectly influence the behavior of the fundamental. An interesting analysis of a model in which the bubble affects the exchange rate both directly and indirectly through the fundamental is analyzed by Miller–Weller [1990a] and by Miller, Weller and Williamson [1989].

Finally, the behavior of the authorities at the edges of the target zone can be summarized as follows: "Reflect when this is sufficient; accommodate when this is necessary.

III. Smooth Pasting a Bubbly Exchange Rate

Consider the case of an exogenous bubble, that is one that does not depend on $\tilde{f}$. The bubble can be either deterministic or stochastic. The graphic representation of this case is shown in Figures 1a and 1b.

We now define the <u>bubble eater</u> of the exchange rate for a given value of the bubble, $B(t)$. The bubble eater of s for $B(t)$ satisfies equation (5) with A_1, A_2 and C determined by equations (6) and (10a–d) for the actually prevailing value of the bubble, that is for $C(t) = -B(t)$. That is the bubble eater of s for $B = \bar{B}$ is given by

$$s = f + \frac{\mu}{\alpha} + \bar{B} + A_1 e^{\lambda_1(f + \bar{B})} + A_2 e^{\lambda_2(f + \bar{B})} \tag{13a}$$

$$s_u = f_u + \frac{\mu}{a} + \bar{B} + A_1 e^{\lambda_1(f_u + \bar{B})} + A_2 e^{\lambda_2(f_u + \bar{B})} \tag{13b}$$

$$0 = 1 + \lambda_1 A_1 e^{\lambda_1(f_u + \bar{B})} + \lambda_2 A_2 e^{\lambda_2(f_u + \bar{B})} \tag{13c}$$

$$s_\ell = f_\ell + \frac{\mu}{\alpha} + \bar{B} + A_1 e^{\lambda_1(f_\ell + \bar{B})} + A_2 e^{\lambda_2(f_\ell + \bar{B})} \tag{13d}$$

$$0 = 1 + \lambda_1 A_1 e^{\lambda_1(f_\ell + \bar{B})} + \lambda_2 A_2 e^{\lambda_2(f_\ell + \bar{B})} \tag{13e}$$

This bubble eater shifts continuously with B; an increase (decrease) in B corresponds to an equal horizontal displacement to the left (right) of the graph. Note that the bubble eater gives a reference trajectory only and does not show the actual movement of f and s when B changes unless $B_u > B(\tau) > B_\ell$ and either $s = s_u$ and $f = f_u$ or $s = s_\ell$ and $f = f_\ell$.

When s is in the interior of the target zone and $B_u > B(\tau) > B_\ell$, its position can be defined with reference to the period τ bubble eater, that is the bubble eater corresponding to the value of B that prevailed at the previous (most recent) visit of s

to one of the boundaries. Let that value of B be B^0. The bubble eater for B^0 can be denoted $s = g(f, B^0) + B^0$. (Without loss of generality we can choose $B^0 = 0$). Then if at time t the actual value of B is B^1, the actual position of s is given by $s = g(f, B^0) + B^1$.

Figure 1a represents the case of a "small" positive bubble, Figure 1b that of a "large" positive bubble. A small positive bubble is one for which equations (10a–d) give a value of f_u greater than s_u, that is $0 < B < B_u$. A large positive bubble has f_u less than or equal to s_u, that is $B \geq B_u$. Analogously, a small negative bubble has f_ℓ less than s_ℓ $(0 > B > B_\ell)$ and a large negative bubble has f_ℓ greater than or equal to s_ℓ $(B \leq B_\ell)$. For sake of brevity, we only consider the case of a positive bubble in detail.

For a small bubble (with $f_u > s_u$), the point $[f_u, s_u]$ (abscissa f_u and coordinate s_u) lies to the right of the 45° line through the origin $s = f$. Since $E_t ds = \alpha(s - f)dt$, any point to the right of the 45° line will be associated with a negative expected rate of change of s, that is with an expected appreciation of the currency. Any point to the left of the 45° line will have a positive expected rate of change of s, that is an expected depreciation of the currency.

In Figure 1a, start from the bubble eater for B^0, given by $s = g(f, B^0) + B^0$, which has a tangency point Ω_0 at the upper boundary and a tangency point Ω_0' at the lower boundary. Now consider what happens when an increase in the size of the bubble to B^1 brings the exchange rate to its upper limit s_u. This can happen for $f = f_u^{*1}$ and $f = f_u^{*2}$, that is at Ω^{*1} and Ω^{*2}. Ω^{*1} and Ω^{*2} lie on the graph $s = g(f, B^0) + B^1$, a vertical distance $B^1 - B^0$ above the bubble eater for B^0.

Note that while Ω^{*1} is to the left of Ω_u (which is on the 45° line through the origin), the bubble eater for B^1, given by $S = g(f, B^1) + B^1$, has a point of tangency to the upper boundary at Ω_1, where $f = f_u^1$, which is to the right of the 45° line. For a sufficiently small bubble, Ω^{*1} can also be to the right of Ω_u (but always to the left of Ω_1). Ω^{*2} will always be to the left of Ω_u. From Ω^{*1} (or Ω^{*2}) there is a discrete intervention (a stock–shift increase in f) which, at a constant exchange rate $s = s_u$, moves the system to Ω_1, the point of tangency of the bubble eater of B^1 and the upper limit of the target zone. The horizontal distance between Ω_0 and Ω_1 is just equal to the difference between B^1 and B^0, i.e. $f_u^0 - f_u^1 = B^1 - B^0$. Moreover, the horizontal distance between f_u^{*1} and f_u^0 is always greater than the distance between Ω_0 and Ω_1. These two distances would be equal only if the slope of the bubble eater were 45° degrees, which is not possible.

Note that at Ω^{*1} the smooth pasting (or no–arbitrage) conditions (10a–d) are not satisfied. At Ω_1 in Figure 1a the smooth pasting conditions are satisfied and the expected rate of change of the exchange rate is negative (Ω_1 is to the right of the 45° line through the origin). Ω_1 is therefore consistent with credible intervention to prevent s from rising above s_u. This is not the case with the large bubble shown in Figure 1b. Ω_1 is to the left of the 45° line through the origin and therefore represents a positive expected rate of change of the exchange rate, inconsistent with a credible defense of the upper limit of the target zone. In this case the intervention rule leads to the following scenario. From Ω_1^* a one–off discrete (stock–shift) increase in f brings the system to Ω_u, that is the point $[s_u^+, s_u]$, an arbitrarily small distance to the right of the intersection of $s = s_u$ and the 45^0 line through the origin. Since s is kept constant at $s = s_u$, there are no arbitrage profits. From Ω_u the system, with A_1 and A_2 now defined by the one–sided 'smooth pasting' conditions given in (11a,b), can now again move into the interior of the target zone.

In Figure 1b, the graph of s with A_1 and A_2 determined by (11a) and (11b) and $B = B_u$, denoted s', coincides with the bubble eater of B_u. For a slightly larger value of B, the graph of s with A_1 and A_2 determined by the one–sided smooth pasting conditions (11a) and (11b) is denoted s''. A larger value of s yields s'''. As B goes to infinity the graph of s approaches a vertical line through Ω_u.

Note that, by construction, we always have a local maximum at the tangency point Ω_u (for finite values of B). For positive unregulated realizations of f, standard infinitesimal reflection stops the regulated value of f from rising beyond s_u^+. For negative realizations of the unregulated f, the system moves (given B) into the interior of the target zone. As in the case of a small bubble, a variation in the value of the bubble in the interior of the target zone represents a vertical displacement of the graph of the exchange rate (with constant values of A_1 and A_2). With a large bubble, the shape of the graph changes every time the exchange rate visits the boundaries at a new value of B.

Since the bubble grows exponentially (in expectation) we expect more frequent interventions at the upper (lower) boundary if the current value of the bubble is positive (negative) than with a zero bubble. We consider these results to be quite intuitive. One would expect an exchange rate subject to a positive and growing (in expectation) bubble to spend a lot of time at the upper boundary of the target zone, with the authorities intervening to counteract not only the unregulated fundamental but also the bubble.

It is apparent that when s reaches s_u with a large bubble, the stock–shift increase in f must take the system to a position strictly to the right of the 45^0 line through the origin. On the 45^0 line the expected rate of depreciation is zero. If we impose the 'smooth pasting' conditions when $f = s = s_u$, (small) positive realizations of increments in the unregulated fundamental will be offset with infinitesimal reflecting interventions. Negative realizations of df would bring the system to the left of the 45^0 line through the origin, that is to a point with a positive expected rate of change of s. This is of course inconsistent with a credible defense of the target zone. If private agents believed that the authorities were stabilizing the exchange rate at $[s_u, s_u]$, a friendly speculative attack of infinitesimal size would occur, bringing the system back to $[s_u, s_u]$. This amounts to stabilizing the fundamental at $f = s_u$. Since

$E_t ds = \alpha(s - f)$ the exchange rate would effectively be expected to be fixed. It is easily seen that an exchange rate cannot remain fixed for any finite interval of time if the fundamental is constant and there is a non–zero speculative bubble. Consider equation (3), for the special case in which f is kept constant at s_u forever. The solution for the exchange rate is given by $s = s_u + B$ which implies that, unless $B = 0$, the exchange rate cannot be constant.

There is a simple, unifying characterization of the interventions that sustain the target zone. It is that the government is credibly committed to infinitesimal reflecting interventions at the boundaries s_ℓ and s_u. The value of the fundamental at which these infinitesimal reflecting interventions at the margins occur are determined by the appropriate two–sided or one–sided "no–arbitrage" (or smooth pasting) boundary conditions.

Consider first the case of a small positive bubble shown in Figure 1a. The value of the fundamental at which the exchange rate reaches, say, the upper boundary (f_u^{*1} in Figure 1a) is different from the value of f that characterizes the tangency of the bubble eater and the upper boundary (f_u^1 in Figure 1a), except in the case of a zero bubble. Since f_u^{*1} is less than f_u^1, the discrete (stock–shift) jump in the fundamental that takes the system to the reflection point Ω_1 can be interpreted as a friendly or sustaining speculative attack which is accommodated by the authorities.

Ω^{*1} is not a credible equilibrium if the private sector knows the government's intervention rule, as there can be no trajectory $s = g(f, B^0) + B^1$ that cuts the upper boundary if there is infinitesimal reflection at the point where $s = g(f, B^0) + B^1 = s_u$. There is a credible reflecting equilibrium at Ω_1. Note that

$$E_t ds\Big|_{f=f_u^{*1}} = \alpha(s_u - f_u^{*1})dt > E_t ds\Big|_{f=f_u^1} = \alpha(s_u - f_u^1)dt.$$

Since the expected rate of change of s at Ω^{*1} in Figure 1a is (positive and) higher than the (negative) expected rate of change of s at Ω^1, the shift from f_u^{*1} to f_u^1 can be interpreted as an increase in the relative demand for home country money due to a decrease in the domestic–foreign interest differential.[6] The authorities obligingly accommodate this friendly or sustaining speculative attack.[7] Note that if at time t the exchange rate reaches s_u at $f = f_u(B(\tau))$, that is when the current value of the bubble is the same as the value of B at the most recent previous boundary visit, the size of the friendly speculative attack is obviously zero, and only a reflecting intervention occurs.

With a large bubble (see Figure 1b) the stock–shift increase in f from $f = f_u^{*1}$ to $f = s_u^+$ can also be interpreted as the passive accommodation of a discrete friendly speculative attack. The bubble eater of the large bubble B^1 in Figure 1b produces a tangency point with the upper boundary at Ω_1, which is to the left of the 45° line through the origin. It therefore has a <u>positive</u> expected rate of change of s, which is inconsistent with a credible defense of the target zone. At Ω_u, the exchange rate is expected to appreciate, which is consistent with a credible target zone.

The policy rule we use to defend the target zone in the presence of rational speculative bubbles is not the only one capable of delivering the survival of the target zone. We could, for instance, apply the one–sided smooth pasting conditions [(11a,b) when $s = s_u$ and (11c,d) when $s = s_\ell$] even when $B_u > B(\tau) > B_\ell$, at any exogenously given value of $f = f_u > s_u$ when $s = s_u$ and at any exogenously given value of $f = f_\ell < s_\ell$ when $s = s_\ell$. We chose the particular rule given in (10a–d), (11a,b) and (11c,d) both because it is quite intuitive and because it is as close as possible to the original model. In particular it includes the traditional 'smooth pasting" solution as the special case when there is no bubble.

IV. A Surfeit of Bubbles

Consider again the general solution to equation (1) given in (3). The first term on the R.H.S. of (3) is commonly called the fundamental solution, s^f

$$(14) \qquad s^f(t) = \alpha \int_t^{\infty} E_t f(v)\, e^{-\alpha(v-t)} dv.$$

Consider equation (15) below, which is a special case of equation (5).

$$(15) \qquad s = f + \frac{\mu}{\alpha} + A_1 e^{\lambda_1 f} + A_2 e^{\lambda_2 f}.$$

In the terminology of McCallum [1983] this is the minimal state solution: it involves only the state variable(s) and the state variables enter in a "minimal" way.

The first two terms on the R.H.S of (15) are the fundamental solution for the unregulated fundamental. All of the R.H.S. of (15), for some given (non–zero) values of A_1 and A_2, is the fundamental solution for the regulated fundamental.

For the unregulated fundamental, non–zero values of A_1 and/or A_2 would permit what Froot and Obstfeld [1989c] call <u>intrinsic</u> bubbles, within the class of functions expressing $s(t)$ as a function of <u>current</u> fundamentals $(f(t))$ <u>only</u>.

For the regulated fundamental, the existence of intrinsic bubbles would require that f enters the solution for the exchange rate in the interior of the target zone through terms other than the ones appearing (with A_1 and A_2 determined by (10a–d), (11a,b) or (11c,d)) on the R.H.S. of equation (14) and with no other variable(s) included. Such intrinsic bubbles are clearly impossible.

The literature has generally cast its discussion of bubbles for our class of models in terms of the behavior of s as a function only of current f for different values of A_1 and A_2. For example, in the case of a freely floating exchange rate, unless $A_1 = 0$, the deviation of s from f will become infinite as f grows without bound (since $\lambda_1 > 0$) and unless $A_2 = 0$, the deviation of s from f will become infinite as f falls without bound (since $\lambda_2 < 0$). $A_1 = A_2 = 0$ rules out only intrinsic bubbles. Many kinds of exchange rate bubbles are possible under a floating rate even if $A_1 = A_2 = 0$. With such bubbles, s can deviate from f (for any given value of f) by eventually unbounded amounts.

It is recognized in the literature (Froot and Obstfeld [1989b] are an example), that the choice of A_1 and A_2 only restricts the relationship between s and f, but that state variables other than f may enter the solution for s through bubbles. In general, bubbles can introduce one or more additional state variables into the solution for s. Subject to the constraint that these bubbles obey (4), they can cause s to deviate in almost any conceivable manner from $g(f, .)$, for any values of A_1 and A_2.

What is not, we believe, recognized as clearly is that s can depend on f in ways other than given by $g(f, \cdot)$ (or the R.H.S. of (15)), provided s also depends (through B) on at least one other state variable. We show below that the bubble at time t, B(t) can be a function of the unregulated fundamental $\bar{f}(t)$, provided it also depends on some other state variable (or state variables) z(t), with $\frac{\partial z(t)}{\partial \bar{f}(t)} = 0$, which ensures that $E_t dB = \alpha B dt$. This implies that within the target zone, dB can be a function of df. z(t) can (but need not) be a function exclusively of past and/or anticipated future values of the unregulated fundamental. We call this class of generalized intrinsic bubbles "fundamental–dependent" bubbles.

(a) Bubbles that don't burst

We begin by considering speculative bubbles, both deterministic and stochastic, that don't collapse. Examples of non–collapsing bubble processes satisfying (4) include the simple exogenous deterministic process given in (16), where $B(0) = B_0$ is an arbitrary initial value for the B process.

$$B(t) = B_0 e^{\alpha t}. \tag{16}$$

An example of a simple exogenous (backward–looking) stochastic bubble satisfying (4) is given in equations (17a,b), with B_0 arbitrary.

$$dB(t) = \alpha B(t)dt + \sigma_b dW_b \tag{17a}$$

$$B(t) = \int_0^t e^{\alpha(t-v)} \sigma_b dW_b(v) + e^{\alpha t} B_0. \tag{17b}$$

The parameter σ_b is the instantaneous variance of the bubble process. Note that σ_b could be a function of s, f, t or any other set of variables without this affecting the fact that equation (17a) satisfies (4). For instance, if σ_B is a positive constant, the bubble can "change sign". With geometric Brownian motion (σ_B proportional to B), sign reversals of the bubble are ruled out, although B can change direction.

Important for our analysis, the bubble process can itself be a function of current, past or anticipated future values of the unregulated fundamental $\bar{f}$. One example is given in equations (18a,b) below. In what follows the variable $z(t)$ is either strictly deterministic or follows a diffusion process. The $z(t)$ process is a twice continuously differentiable function of its arguments. The key defining property of $z(t)$ is that it does not vary with $\bar{f}(t)$, i.e. $\frac{\partial z(t)}{\partial \bar{f}(t)} = 0$. For the example given below, the z process given in equation (18b) (a backward–looking deterministic process) generates, together with equation (18a), a bubble process that satisfies equation (4).

$$(18a) \qquad B(t) = k\bar{f}(t) + z(t) \qquad k \neq -1$$

$$(18b) \qquad z(t) = \int_0^t e^{\alpha(t-v)}[k(\alpha\bar{f}(v) - \mu)]dv + e^{\alpha t}z_0$$

z_0 is an arbitrary initial value for the z process. Note that the boundary conditions (10a–d) cannot be satisfied if $k = -1$ and that, given the boundary conditions (11a,b) or (11c,d), A_1 and A_2 are indeterminate if $k = -1$.

More generally, equation (18a) will hold provided z satisfies

$$(19) \qquad E_t dz(t) = [k(\alpha\bar{f}(t) - \mu) + \alpha z(t)]dt.$$

The z process given in (19) (and therefore the B process itself) can be forward–looking, backward–looking (or a linear combination of the forward–looking and backward–looking solutions), it can be strictly deterministic or stochastic.

An example of a forward–looking solution for z (now treated as a non–predetermined variable) which satisfies equations (18a) and (19) (and of course equation (4)) is given in equation (20).

$$(20) \qquad z(t) = -\int_t^{\infty} e^{\alpha(t-v)}E_t k(\alpha\bar{f}(v) - \mu)dv + R(t).$$

Here $R(t)$ is any strictly deterministic or stochastic process which satisfies $E_t dR(t)dt = \alpha R(t)dt$. It can be predetermined or non–predetermined. One possible choice for $R(t)$ would be the process given on the right–hand side of (18b). This would make the bubble $B(t)$ in (18a) a function of current, past and anticipated future values of the unregulated fundamental.

Another interesting bubble process involving $\bar{f}$ is the following:

$$\text{(21a)} \qquad B(t) = \bar{f}(t)z(t)$$

$$\text{(21b)} \qquad \begin{cases} dz(t) = \bar{f}(t)^{-1}[\alpha\bar{f}(t) - \mu]z(t)dt + \sigma_z dW_z & \bar{f} \neq 0 \\ z(t) = 0 & \bar{f} = 0 \end{cases}$$

(dW_z and dW_f are assumed to be contemporaneously independent).

An example of a class of rational bubbles that has the property that $\frac{\partial B(t)}{\partial f(t)}$ depends on $\bar{f}(t)$ is given in (22a–b).

$$\text{(22a)} \qquad B(t) = \bar{f}(t)^2 + z(t)$$

$$\text{(22b)} \qquad E_t dz(t) = [\alpha(\bar{f}(t)^2 + z(t)) - 2\bar{f}(t)\mu - \sigma_f^2]dt.$$

A simple backward–looking deterministic solution for z that satisfies (22b) is

$$\text{(23)} \qquad z(t) = \int_0^t e^{\alpha(t-v)}[\alpha\bar{f}(v)^2 - 2\bar{f}(v)\mu - \sigma_f^2]dt + e^{\alpha t}z_0.$$

If B can depend on the regulated value of the fundamental as well as on the unregulated fundamental, we can generate the very unusual bubble given in equation (24).

$$\text{(24)} \qquad B(t) = -g(f(t), B(\tau)) + k\bar{f}(t) + z(t).$$

This bubble is consistent with the law of motion (4), provided that (25) holds:

$$(25)\quad E_t dz(t) = \{-\alpha[g(f(t),.) - k\bar{f}(t) - z(t)] + [g'(f(t),.) - k]\mu + \tfrac{1}{2}g''(f(t),.)\sigma_f^2\}dt$$

Equation (26) is an example of a simple stochastic process for z that satisfies (25)

$$(26)\qquad z(t) = \int_0^t e^{\alpha(t-v)}\Big[\{-\alpha[g(f(v),.) - k\bar{f}(v)] + [g'(f(v),.)-k]\mu + \tfrac{1}{2}g''(f(v),.)\sigma_f^2\}dv + \sigma_z dW_z(v)\Big]dv + e^{\alpha t}z_0.$$

While this bubble is consistent with the law of motion (4) within the target zone, it does not permit the boundary conditions to be satisfied. With this process for $B(t)$ the exchange rate is given by

$$(27)\qquad s(t) = k\bar{f}(t) + z(t).$$

Note that in this case s increases or decreases linearly with $\bar{f}$ (if $k \neq 0$), thus making "smooth pasting" boundary conditions impossible to apply. When $k = 0$, $s(t)$ is independent of $\bar{f}(t)$, although $z(t)$ of course depends on past values of $\bar{f}$ (and of f).

(b) Bubbles that burst

Blanchard [1979] provides an interesting example of a rational speculative bubble whose expected duration (time until the moment a collapse occurs) is finite. It is most easily motivated in discrete time. Equation (28) is the discrete analog of equation (1).

$$(28)\qquad s_t = \alpha(1+\alpha)^{-1}f_t + (1+\alpha)^{-1}E_t s_{t+1}. \quad \alpha > 0$$

The solution of (28) conditional on current and expected future values of f_t is

$$s_t = \alpha(1+\alpha)^{-1} \sum_{i=0}^{\infty} (1+\alpha)^{-i} E_t f_{t+i} + B_t \tag{29}$$

where B_t satisfies

$$E_t B_{t+1} = (1+\alpha) B_t. \tag{30}$$

Consider the following process for B

(31) $B_{t+1} = (1-\pi)^{-1}(1+\alpha)B_t + \epsilon_{t+1}$ with probability $1-\pi$

$B_{t+1} = \hat{\epsilon}_{t+1}$ with probability π

where $0 \le \pi \le 1$ and

$$E_t \epsilon_{t+1} = E_t \hat{\epsilon}_{t+1} = 0. \tag{32}$$

It is easily checked that (31) and (32) satisfy (30). Equations (31) and (32) define a bubble (stochastic if ϵ_t or $\hat{\epsilon}_t$ is a random variable) for which there is a constant probability of collapse, π, each period. If a collapse occurs in period t+1, the exchange rate returns to its fundamental value if $\hat{\epsilon}_{t+1} = 0$. More generally, if $\hat{\epsilon}_{t+1}$ is random, the expected post–collapse value of the exchange rate is its fundamental value, but the realized value of the exchange rate can differ from this expectation by a zero mean forecast error. If the bubble collapses in period t+1 and

$\hat{\epsilon}_{t+1}$ is non–zero, a new bubble starts in period $t+1$ which follows a law of motion like (31) and (32), but possibly with a different value of π and different (although still zero mean) random disturbance terms ϵ and $\hat{\epsilon}$.

In continuous time the same idea is captured as follows. π now is the constant instantaneous probability of collapse of the bubble. Conditional on the bubble not having collapsed by time t, the probability of the bubble lasting till time $t + \Delta t$ is $e^{-\pi\Delta t}$ $(\Delta t \geq 0)$. We consider the following process for B. As before, W_b and $\hat{W}_b$ are standard Wiener processes. Conditional on the bubble having lasted till time t, we have

$$B(t+\Delta t) = e^{(\alpha+\pi)\Delta t}B(t) + \sigma_b \int_t^{t+\Delta t} e^{(\alpha+\pi)(t+\Delta t-v)}dW_b(v) \tag{33}$$

with probability $e^{-\pi\Delta t}$

$$B(t+\Delta t) = \hat{\sigma}_b \int_t^{t+\Delta t} e^{(\alpha+\pi)(t+\Delta t-v)}d\hat{W}_b(v)$$

with probability $1 - e^{-\pi\Delta t}$.

It follows that the expected rate of change of B per unit time over the interval $[t, t+\Delta t]$ is

$$\frac{E_tB(t+\Delta t) - B(t)}{\Delta t} = \frac{(e^{\alpha\Delta t} - 1)}{\Delta t}B(t). \tag{34}$$

In the limit as $\Delta t \to 0$ this becomes $E_tdB(t) = \alpha B(t)dt$, as required. While the bubble lasts, the instantaneous rate of change of the bubble is given by

$$dB(t) = (\alpha + \pi)B(t)dt + \sigma_b dW_b(t).$$

While the expected rate of change of the bubble is $E_t dB = \alpha B dt$, the expected rate of variation of B conditional on the bubble surviving the next instant, is given by $(\alpha + \pi)Bdt$.

Let t_c be the (random) time when the bubble collapses. If the value of B at the time of collapse, $B(t_c) = \hat{\sigma}_b \int_t^{t_c} e^{(\alpha+\pi)(t_c - v)} d\hat{W}_b(v)$ is non–zero, a new bubble will start at t_c, possibly with a different value of π, different white noise processes dW_b and $d\hat{W}_b$ and different values for σ_b and $\hat{\sigma}_b$.

A Blanchard bubble is perfectly compatible with our target zone model. When it collapses there will be a discrete change in the level of the exchange rate, which lies on the trajectory appropriate to the new (post–collapse) bubble. Since this change is unexpected, no arbitrage opportunities arise.

(c) Target zones with fundamental–dependent bubbles

If the value of the bubble at time t is a function of the contemporaneous value of the unregulated fundamental, then within the target zone the response of the exchange rate to the fundamental will be affected relative to the case of an exogenous bubble (and, a–fortiori, relative to the no–bubble case). This is straightforward, since in this case $\frac{\partial \bar{f}}{\partial f} = 1$. As long as $\frac{\partial B(t)}{\partial f(t)} \neq -1$, the tangency conditions (10c–d), (11b) and (11d) are the same as in the case of an exogenous bubble.

V. The Distribution Function of the Exchange Rate in a Target Zone in the Presence of a Rational Speculative Bubble

By Ito's lemma, the exchange rate in the target zone is a diffusion process with stochastic differential

$$(35) \qquad ds(t) = [g_f(f,.)\mu + \tfrac{1}{2}g_{ff}(f,.)\sigma_f^2]dt + g_f(f,.)\sigma_f dW_f + dB.$$

Let $Var_t x(t')$ denote the variance of $x(t')$ conditional on the information available at time t, that is $Var_t x(t') \equiv E_t[x(t') - E_t x(t')]^2$. The instantaneous variance of the bubble process is given by $\sigma_B^2(f, \bar{f}, s, B, \cdot)$. Since the only restriction on the bubble process is that $E_t dB = \alpha B dt$, it is certainly permissible for the instantaneous variance of the bubble process to depend on f, $\bar{f}$, s, B or othcr variables.

As before, recognizing that B may depend on $\bar{f}$, we write

$$B = b(\bar{f},z) \text{ with } \frac{\partial z}{\partial \bar{f}} \equiv 0.$$

Using self–explanatory notation we obtain

$$(36a) \qquad E_t ds(t) = (g_f+b_f)\mu + b_z E_t dz(t) + \tfrac{1}{2}\Big[(g_{ff}+b_{ff})\sigma_f^2 + b_{zz}\sigma_z^2 + 2b_{zf}\rho_{fz}\sigma_f\sigma_z\Big]dt$$

and

$$(36b) \qquad Var_t ds(t) = \Big[(g_f+b_f)^2\sigma_f^2 + b_z^2\sigma_z^2 + 2(g_f + b_f)b_z\rho_{fz}\sigma_f\sigma_z\Big]dt.$$

For a bubble process to be consistent with a target zone under the specified intervention rule, it must be the case that, for any size bubble, the boundary conditions characterize tangency points that are local maxima (minima) if the tangency is at the upper (lower) boundary. For a bubble that is consistent with the law of motion in (4) but does not permit smooth pasting according to (10a,d), we refer to the bubble in (18a,b) with $k = -1$ or the bubble given in (24) and (25).

Note that the instantaneous conditional variance of the change in the exchange rate no longer in general goes to zero as the boundaries of the target zone are approached. As the exchange rate approaches the boundary, the variance of the exogenous component of the bubble need not go to zero nor need $g_f + b_f$ tend to zero.

If, in the spirit of Obstfeld–Rogoff and Diba–Grossman, we accept the proposition that under free floating bubbles cannot occur, the conditional expectation and variance of the change in the exchange rate with a free float are (letting $\hat{s}$ denote the exchange rate under a free float):

$$E_t \frac{d\hat{s}(t)}{dt} = \mu$$

and

$$Var_t \frac{d\hat{s}(t)}{dt} = \sigma_f^2.$$

Trivially, since bubbles can exist in a target zone, the instantaneous conditional variance of changes in the exchange rate may be less under free floating than under a target zone. If uncovered interest parity holds, it must be the case that the interest differential δ is given by

$$\delta(t) \equiv i(t) - i^*(t) = E_t\, ds(t)/dt.$$

It follows that the instantaneous conditional mean and variance of the change in the interest differential in the target zone are given by

(37a) $$E_t \frac{d\delta(t)}{dt} = \alpha\left[(g_f+b_f-1)\mu+b_z F_t dz + \frac{1}{2}((g_{ff}+b_{ff})\sigma_f^2+b_{zz}\sigma_z^2+2b_{zf}\rho_{fz}\sigma_f\sigma_z)\right]$$

(37b) $$Var_t \frac{d\delta(t)}{dt} = [\alpha(g_f+b_f-1)]^2\sigma_f^2 + (\alpha b_z)^2\sigma_z^2 + 2\alpha^2(g_f+b_f-1)b_z\rho_{fz}\sigma_f\sigma_z.$$

As noted in Svensson [1989], if $\sigma_z = 0$ (and therefore also $\rho_{fz} = 0$) equations (36b) and (37b) imply that (denoting the conditional standard deviation by SD_t)

$$SD_t(\frac{ds}{dt}) - \alpha^{-1}SD_t(\frac{d\delta}{dt}) = \sigma_f.$$

This finding of a constant trade–off between the instantaneous variability of the change in the exchange rate and the instantaneous variability of the change in the interest differential is no longer automatically valid when there is a bubble, as can be verified by inspection of (36b) and (37b).

The conditional instantaneous mean and variance of the change in the interest differential under a free float, $d\hat{\delta}$, are:

$$E_t d\hat{\delta}(t) = 0$$

and

$$Var_t d\hat{\delta}(t) = 0.$$

With or without a bubble, the free float therefore always delivers more stability in changes in the interest differential than does the target zone.

The relationship between the level of the exchange rate and the interest differential in the target zone is found by noting that

$$\frac{\partial\delta}{\partial f} = \alpha(g_f + \frac{\partial B}{\partial f} - 1)$$

and

$$\frac{\partial s}{\partial f} = g_f + \frac{\partial B}{\partial f}.$$

It follows that

$$\frac{\partial s}{\partial \delta} = \frac{g_f + \frac{\partial B}{\partial f}}{\alpha(g_f + \frac{\partial B}{\partial f} - 1)}.$$

When there is no bubble and the range of permissible values of f is $f_\ell \leq f \leq f_u$ (that is the values of f corresponding to the upward–sloping part of the bubble eater for $B = 0$) we have, since $0 \leq g_f < 1$ for $f_\ell \leq f \leq f_u$, a downward–sloping relationship between the level of the exchange rate and the interest differential. This relationship is qualitatively unaffected by the bubble if the current value of the bubble is independent of the current value of the fundamental, if we again restrict the analysis to the upward–sloping part of the $h(f,B(\tau), B(t)) = g(f,.) + B(t)$ function. There are, however, no good reasons for this restriction when there are bubbles, so the relationship between the level of the exchange rate and the interest differential can change sign. This is shown in Figure 2 for an exogenous bubble.

Harrison [1985, pp. 89–92] shows that the steady state or unconditional density function π of Brownian motion f with instantaneous variance σ_f^2 and drift μ which is regulated on the close interval $[f_\ell, f_u]$ is given by:

$$\pi(f) = [f_u - f_\ell]^{-1} \qquad \text{if } \mu = 0 \text{ (the uniform density)}$$

$$\pi(f) = \frac{\theta e^{\theta f}}{e^{\theta f_u} - e^{\theta f_\ell}} \qquad \text{if } \mu \neq 0 \text{ (truncated exponential density)}$$

$$\theta = 2\mu\sigma_f^{-2}.$$

Without a bubble it follows that, since $s = g(f)$ is a strictly increasing function of f with continuous first derivative on the open interval (f_ℓ, f_u) and the implicit (inverse) function $f = g^{-1}(s)$ therefore exists on the target zone (except at the end points), the steady state density function of s, $\varphi(s)$ say, is given by:

$$\varphi(s) = \frac{\pi(g^{-1}(s))}{g'(g^{-1}(s))} \qquad s_\ell < s < s_u.$$

Svensson [1989] shows that the resulting distribution of exchange rates is U–shaped.

With a bubble, however, the long–run behavior of the exchange rate is quite different. First of all, one of the state variables, the bubble, is non–stationary and unregulated. It therefore does not possess a steady state distribution. The other state variable, f, is of course also non–stationary when unregulated, but regulation may still ensure that it has a steady state distribution. Note, however that bounds set by the controller are bounds on s, not on f. It is clearly possible for f to become a negative number smaller than the value of f_ℓ associated with $B = 0$ (for instance if a positive bubble grows steadily and there is a long sequence of, on average, negative realizations of df within the target zone). Similarly, f can become a positive number larger than the value of f_u associated with $B = 0$ if a negative bubble steadily grows in size and there is a long run of, on average, positive realizations of df.[8]

While the derivation of the steady–state distribution of s for most interesting cases (including the exogenous stochastic bubble and the stochastic bubble that depends on f) remains to be done, a few general remarks can be made.

Consider first the case of an exogenous, non–stochastic bubble. The asymptotic distribution of the exchange rate will, if the bubble is positive, put all its probability mass near its upper boundary value s_u. In this very simple case, the target zone doesn't kill the bubble, but the bubble effectively kills the target zone and turns it, asymptotically, into a fixed exchange rate regime.

Now consider the case where the bubble is stochastic. If we have the exogenous geometric bubble $dB = \alpha B dt + \sigma_B B dW_B$, the bubble cannot change sign, although it can change direction. Take the case with $B(0) > 0$. If the bubble ever reaches $B = 0$, it dies and we are back henceforth in the case studied by Svensson. Clearly we expect to have a non–degenerate distribution of s in this case, but still with more mass at s_u than without a bubble.

If the exogenous bubble process is given instead by $dB = \alpha B dt + \sigma_B dW_B$ with $\sigma_B > 0$ but constant, B will, in a representative history, assume both positive and negative values. The bubble never dies in this case. Relative to the case studied by Svensson we would expect the long–run density function to have more mass at the two end points but with no particular bias towards either s_u or s_ℓ. Asymptotically, the influence of the initial conditions for B on the density function for s should vanish. We hope to back up these intuitions with solid analysis in the near future.

VI. Bubbles, target zones and intervention rules: what gives when they are inconsistent?

Our view of what happens to speculative bubbles under a credible target zone is rather different from that of Miller, Weller and Williamson [1989] (henceforth MWW). The formal analysis in their paper is cast in terms of Blanchard bubbles, but would apply a–fortiori to the non–collapsing bubbles we considered. MWW put their argument succinctly:

> *If all market participants believe that, when the exchange rate hits the edge of the band, the authorities will defend the rate by sudden intervention (designed to produce a jump in the rate), this must cause the bubble to burst. But this is*

> *inconsistent with the existence of a bubble in the first place: If all know this collapse is certain to occur at time t, all wish to sell the currency at t–ε. But then collapse will occur at t–ε. Repeating this argument we find that collapse must occur at time zero that is, all such bubbles are "strangled at birth".*
>
> (*Miller, Weller and Williamson* [*1989*]).

It should be recognized that in this paper we have changed the intervention rules followed by the authorities from what was assumed by MWW and by the other contributors to this literature. In addition to the standard reflection policies, we allow the accommodation by the authorities of sustaining speculative attacks. The intervention policies considered by MWW and in the rest of this literature, do not have this feature. It should therefore come as no surprise that from different assumptions we reach different conclusions.

Given the assumption that the only form of intervention is regulation at the boundaries through infinitesimal reflecting operations, the conclusion that this configuration is inconsistent with rational bubbles follows. One common response to this inconsistency is to treat the bubble as a *residual.* The reasoning goes as follows: at a point like Ω^{*1} in Figure 1a there would be a discontinuous anticipated fall in the value of s from s_u to $\hat{s}$, the value of the exchange rate at $f = f_u^{*1}$ on the bubble eater for $B = 0$, given by $s = g(f, B^0) + B^0$, interpreting B^0 to equal zero. This violates the no–arbitrage condition.

What is presented in the previous paragraph as a "real–time" event, the *fall* of the exchange rate from s_u to $\hat{s}$, can in fact be no more than the designation of $\hat{s}$ as the only value of the exchange rate reached when $f = f_u^{*1}$, that is consistent with the intervention rule. Equivalently, $B \equiv 0$ is the only value of the bubble consistent with the intervention rule. If the system really were (somehow) at Ω^{*1}, there would be an inconsistency. We cannot deduce how the exchange rate would behave starting from an inconsistent position.

Note that it is not just the assumption of a credible exchange rate target zone that produces the inconsistency, but the combination of a credible target zone and a particular intervention rule that can only support the target zone if no bubble exists. Our alternative intervention rule, which has a discrete (stock–shift) increase in f at Ω^{*1} from f_u^{*1} to f_u^1 and a corresponding movement of the equilibrium from Ω^{*1} to Ω_1, consistently combines a credible target zone and rational bubbles.

An obvious question would seem to be: Why would the authorities adopt a rule that permits the presence of bubbles rather than a rule that precludes their existence?

Putting the question this way prejudges the answer, because it assumes that, given an inconsistency between having a credible target zone, a rational bubble and a particular intervention policy, the inconsistency <u>must</u> be resolved by dropping the bubble, rather than by dropping the target zone or the intervention policy. We prefer to view the existence of a bubble as a <u>datum</u>: A bubble either exists or it doesn't, and the authorities must design their intervention policy accordingly, if they wish to support the target zone.

A particular intervention rule (e.g. infinitesimal reflecting interventions at the boundaries without accommodation of friendly speculative attacks) may of course be inconsistent with having a credible target zone in the presence of bubbles. This implies, in our view, a collapse either of the target zone or of the intervention rule, but not of the bubble, which is not an object of choice at some initial date, not even a collective one. Given the existence of a bubble, a credible target zone requires a policy rule specification that permits the bubble and the target zone to coexist. Of course it must also be capable of handling the no–bubble case . Our rule is an example of such a consistent policy.[9]

VII. Conclusions

The theory of exchange rate behavior within a target zone as developed in the recent literature holds that exchange rates under a currency band regime are less responsive to fundamental shocks than exchange rates under free float, provided that the intervention rules of the Central Bank(s) are common knowledge. Moreover, there always exists a trade–off between the instantaneous volatility of changes in the exchange rates and changes in the interest rate differential, independently of the size of the band and the degree of credibility of the target zone. These results are derived after having assumed *a priori* that "rational excess volatility" (due to so called *rational bubbles*) does not occur in the foreign exchange market. Implicitly or explicitly, it is assumed that speculative bubbles are incompatible with the existence and the persistence of a credible target zone, so that they never materialize.

We consider instead a setup in which the existence of speculative behavior is a *datum* the Central Bank has to deal with. We show that there is no incompatibility between the existence of a target zone and the presence of rational bubbles. Rather, there are intervention rules that should be followed by the Central Bank when speculative bubbles arise, and these same rules include as a special case the traditional policies for defending an exchange rate band when speculative bubbles do not occur.

In the standard model, the defense of a target zone is guaranteed by intermittent variations in domestic credit (say through open market operations) and/or non sterilized foreign market interventions that take place when the exchange rate hits one of the limits of the band. For instance, when the exchange rate reaches the upper boundary, the stock of foreign reserves (or the stock of domestic credit) is reduced in order to prevent the exchange rate from raising further. Heuristically, these operations can be characterized as infinitesimal *reflecting* interventions. Although the size of the reflecting operations may be quantitatively limited, the induced expectations stabilize the exchange rate even before the upper or lower limit is reached.

We show that in the *cum bubble* setup reflecting interventions are insufficient. In fact, when speculative bubbles arise, the companion phenomena of speculative attacks on the target zone must occur as well. These speculative attacks are stock–shift reshuffles of private agents' financial portfolios led by rational expectations of changes in the rate of exchange rate depreciation. As an example, if the rate of appreciation of the exchange rate were suddenly expected to increase, rational private agents would increase their demand for home country money and/or decrease their demand for foreign country money due to a decrease in the domestic–foreign interest rate differential.

We show that the defense of the target zone in the presence of bubbles is guaranteed if the Central Bank *accommodates* speculative attacks when the latter are *friendly*, that is when they are consistent with the survival of the target zone itself (given the Central Bank's rule). This intuitive policy rule is compatible with self–fulfilling expectations: a friendly attack occurs only if agents know that it will be accommodated, and exactly because of the passive accommodation the speculative attack *sustains* the target zone. Hence, the policy rule is summarized by the maxim "Reflect when this is sufficient; accommodate when this is necessary".

We show that when the exchange rate hits one of the boundaries with a non–zero bubble, an accommodated friendly speculative attack occurs, followed by reflecting interventions.

Many of the conclusions reached in the existing literature do not appear sufficiently robust when speculative bubbles are considered as well. First, it is not true anymore that the instantaneous volatility of exchange rates within a target zone is always less than the instantaneous volatility of exchange rate under free float, provided that, for familiar reasons, in free float no speculative bubble arises. Second, the finding of a constant trade–off between the instantaneous variability of the change in the exchange rate and the instantaneous variability of the change in the interest rate differential is no longer automatically valid when there is a bubble.

Moreover, the standard theory characterizes a stable relation between the level of the exchange rate and the expected rate of depreciation (equal to the interest rate differential if uncovered interest parity holds). According to this model, the higher the exchange rate, the lower the interest rate differential. Moreover the interest rate differential is always negative in the neighborhood of the upper boundary of the exchange rate band; the opposite result holds in the neighborhood of the lower boundary. We show that this relation is no longer stable when bubbles arise.

NOTES

[1]Consider a dynamic linear rational expectations model with constant coefficients that has the usual saddlepoint configuration (as many predetermined variables, n_1 say, as stable characteristic roots and as many non–predetermined variables, n_2 say, as unstable characteristic roots). Transform the system to canonical variables, by diagonalizing it or by using Jordan's canonical form. Group together the n_2 dynamic equations containing the canonical variables whose homogeneous equations are governed by the unstable roots. The general solution for these non–predetermined canonical variables can contain an n_2–dimensional bubble process that must satisfy the homogeneous equation system of the n_2 non–predetermined canonical variables. Bubble processes will therefore be non–stationary (in expectation). The state variables of the model, which are linear combinations of the canonical variables, will, if there is a bubble, be non–stationary. If there are more non–predetermined state variables than unstable roots, stationary bubbles will of course be possible even in linear models.

[2]Similar arguments can be made for non–stationary bubbles (and the non–stationary behavior of key endogenous variables frequently associated with non–stationary bubbles) in certain non–linear models. For instance, in overlapping generations (OLG) models of a competitive economy with a single perishable commodity and with non–interest–bearing outside money as the only store of value, rational deflationary bubbles (i.e. non–stationary bubbles with the price level declining to zero) have been shown to be infeasible. Take for instance the case of a two–period OLG model with a constant population in which the nominal money stock is constant,

all money is held by the old and only the young receive a constant bounded endowment stream of the single commodity. If the price level were to fall without bound, real cash balances would be increasing without bound and the demand for the commodity by the old generation would increase without bound. Since the supply of the good each period is bounded above by the sum of the endowments of the young, sooner or later the demand for the commodity must outstrip the supply. A sequence of money prices falling to zero can therefore not be rational expectations equilibrium (see e.g. Hahn [1982, p. 10]).

Unless very strong restrictions are placed on the private utility functions, rational inflationary bubbles (with the price level increasing without bound despite a constant nominal money stock) can exist in such an economy. The sequence of rising prices would, in the limit, drive the real value of money to zero. The steady state to which such a model converges is that of a non–monetary economy. To rule out inflationary bubbles as well, Obstfeld and Rogoff [1983], in an infinite–lived representative agent model, imposed a political–technological restriction on the terminal value of money: the government fractionally backs the currency by guaranteeing a minimal real redemption value for money. Even if private agents are not completely certain that they can redeem their money in any given period, this suffices to rule out speculative hyperinflations. While this assumption on government behavior seems quite ad–hoc, it is often cited as the second blade of the scissors that cut the lifeline of non–stationary rational speculative bubbles, both deflationary and inflationary. Diba and Grossman [1988] also derive sufficient conditions for ruling out non–stationary inflationary bubbles.

[3] Sargent and Wallace's "Unpleasant Monetarist Arithmetic" model (Sargent and Wallace [1981]) can exhibit stationary rational bubbles. Even a first order deterministic non–linear difference equation may exhibit various kinds of periodic

solutions or chaotic behavior (see e.g. Benhabib and Day [1981, 1982] and Grandmont [1985]). Stationary bubbles are easily generated by such models (Azariadis [1981], Azariadis and Guesnerie [1986], Chiappori and Guesnerie [1988], Woodford [1987], Farmer and Woodford [1986]). Second order deterministic non–linear differential equations can generate limit cycles and third order non–linear differential equations can exhibit chaotic behavior. Again, such models can support stationary rational speculative bubbles.

[4]This interpretation of f comes from the two–country mini–model outlined below. All variables except interest rates are in natural logarithms. m is the home country nominal money stock, p the home country price level, y home country real GDP, i the home country short nominal interest rate and s the spot price of foreign exchange. Corresponding foreign country variables are starred.

$$m - p = ky - \lambda i, \quad k, \lambda > 0 \qquad \text{(Home monetary equilibrium)}$$

$$m^* - p^* = ky^* - \lambda i^* \qquad \text{(Foreign monetary equilibrium)}$$

$$p = p^* + s \qquad \text{(Purchasing Power Parity)}$$

$$E_t ds(t) = [i(t) - i^*(t)]dt \qquad \text{(Uncovered Interest Parity)}$$

From this very simple model we obtain the following relation:

$$s(t)dt = [m(t) - m^*(t) - k(y(t) - y^*(t))]dt + \lambda E_t ds(t).$$

This corresponds to equation (1) with $\lambda = \alpha^{-1}$ (the interest semi–elasticity of money demand) and $f = m - m^* - k(y - y^*)$.

[5]Equation (5) is of course equivalent to the perhaps more familiar form

$$s(t) = f(t) + B(t) + \frac{\mu}{\alpha} + \bar{A}_1(t)\, e^{\lambda_1 f(t)} + \bar{A}_2(t)\, e^{\lambda_2 f(t)}$$

where $\bar{A}_1$ and $\bar{A}_2$ are constant as long as s is in the interior of the target zone, but can change when s is at one of its boundaries. For many bubbles, including the exogenous ones, the representation given in (5) is attractive because it can bring out clearly the horizontal shift of the *bubble eater* (defined below) when the magnitude of the bubble varies.

[6]While the expected rate of change of s at the reflection points on the upper boundary (points like Ω_1 or Ω_u) is always less than that at points like Ω^{*1}, the expected rate of change at Ω^{*1} can, for very small bubbles, be negative.

[7]The notion of a friendly or sustaining speculative attack is related to the "sustaining" money demand by arbitrageurs in one of the solutions to the "gold standard paradox" proposed by Buiter and Grilli [1989]. See also Krugman and Rotemberg [1990].

[8]The infimum of f is where the downward–sloping part of the bubble eater for B_u cuts the upper boundary. The supremum of f is where the downward–sloping part of the bubble eater for B_ℓ cuts the lower boundary.

[9]A separate argument that might be made against bubble equilibria takes aim

at its most striking feature: the possibility of very frequent interventions at the edges of the zone.

Consider for simplicity a deterministic bubble that starts from a positive initial value. It might be argued that the interventions in the fundamental that are required to offset such a non–stationary bubble are not sustainable because, as the magnitude of the bubble grows over time, interventions at the upper boundary can be expected to become more and more frequent. Finite international reserves (assuming these are the intervention medium) are bound to be exhausted in due course.

It should be noted that a very similar argument can be made even if there is no bubble, as long as there is positive drift in the fundamental. While the expected value of the fundamental grows linearly (at a constant rate μ) rather than exponentially as the bubble does, the drift of the unregulated fundamental will also in finite time cause any finite stock of reserves to be exhausted with probability one. Indeed, even without drift in the fundamental (and without a bubble), a stochastic fundamental process driven by Brownian motion will bring any finite stock of reserves down to any positive lower bound in finite time with probability one (see Buiter [1989]).

The problems of international reserve exhaustion created by the bubble are of course less severe if the bubble is a Blanchard bubble, which has finite expected duration. Note also that in the most common interpretation of our model, the fundamental f stands for relative home country money supply minus relative real income – related money demand. Taking relative nominal money stocks as the object of regulation, there is nothing in the logic of the model that requires international reserves to be used to regulate the money stocks. Domestic credit expansion (whether reflecting open market operations or monetary financing of government budget deficits) can achieve the same monetary objectives.

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