The Fallacy of the Fiscal Theory of the Price Level – Once More

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Abstract

The fiscal theory of the price level fails to meet the necessary conditions for valid dynamic equilibrium analysis: (1) the number of equations equals the number of unknowns; the number of state variables equals the number of first-order differential or difference equations; there are as many state variables as boundary conditions; (2), if (1) holds, the model has one or more solutions; (3) if (1) and (2) hold, these solutions make economic sense. The fundamental fallacy of the FTPL is its treatment of the intertemporal budget constraint (IBC) of the State. *Either* the fiscal-financial-monetary program (FFMP) of the State satisfies its IBC identically (the State pursues a Ricardian FFMP) and its debt always trades at its contractual value, *or* it pursues an arbitrary, non-Ricardian FFMP. It that case its IBC may be satisfied (with equality) in equilibrium and the IBC of the State can be treated as a bond pricing equilibrium equation, but with no guarantee that the market price of the debt is equal to its contractual value. The State can waste fiscal space or be in default.

The FTPL instead asserts that arbitrary, non-Ricardian policies for public spending, taxation, interest rates and monetary issuance need satisfy the IBC *in equilibrium* only, but that the sovereign debt will still always be priced at its contractual value. This miracle happens because either the price level (in the original FTPL) or current and anticipated future nominal and real discount factors, (in the New-Keynesian version of the FTPL developed by Sims), adjust to make the real contractual value of the outstanding stock of nominal public debt equal to the present discounted value of current and future real primary surpluses plus seigniorage.

In reality this means overdetermined or inconsistent systems unless (1) the price level is flexible, (2) the interest rate is the monetary policy instrument and (3) there is a non-zero stock of nominal government bonds. Thus, when the nominal money stock is the policy instrument, there is overdeterminacy. A sticky price level implies either overdeterminacy or another kind of inconsistency. When all three conditions are satisfied, there are unacceptable anomalies: negative price levels can occur; the FTPL can price money when money does not exist except as a numeraire; the logic of the FTPL applies equally to the IBC of an individual household; when the equilibrium bond pricing equation is specified correctly, there is no FTPL; there is no FTPL unless there are nominal government bonds.

The FTPL is not about monetary vs. fiscal dominance or active v. passive fiscal policy.

The FTPL implies government debt is never a problem; the price level or the level of real economic activity ensure sovereign solvency - and not through unanticipated inflation or financial repression. If acted upon by fiscal authorities, the consequences could be severe.

There is a correct fiscal theory of seigniorage. The issuance of return-dominated and/or irredeemable central bank money creates fiscal space and ensures that a combined monetary-fiscal stimulus can always boost nominal aggregate demand.

Keywords: Fiscal theory of the price level; intertemporal budget constraint; equilibrium bond pricing equation; monetary and fiscal policy coordination.
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1. Introduction

The fiscal theory of the price level (FTPL) is making an unexpected and undesirable comeback. On 1 April, 2016, a conference with as its theme "Next Steps for the Fiscal Theory of the Price Level" was held at the Becker Friedman Institute for Research on Economics at the University of Chicago.[2] Many of the originators of the FTPL participated and asserted its continued validity and relevance, including Christopher Sims, John Cochrane and Eric Leeper (see e.g. Sims (2016b), Cochrane (2016b, c) and Jacobson, Leeper and Preston (2016)).[3]

I argue in this paper that there is a good fiscal theory of the price level, or rather a fiscal theory of seigniorage (FTS), and a bad fiscal theory of the price level. The "Next Steps …" conference promoted the bad FTPL – a false theory based on an elementary but fundamental fallacy: it confuses the intertemporal budget constraint (IBC) of the State with a misspecified nominal sovereign bond pricing equilibrium condition. In a properly specified (Standard) monetary dynamic general equilibrium model *either* the State ensures that its fiscal-financial-monetary program (FFMP) satisfies its IBC identically (in this case the State is said to pursue a Ricardian FFMP) *or* it pursues an arbitrary, non-Ricardian FFMP that is not guaranteed to satisfy its IBC identically. When the State pursues a Ricardian FFMP, its debt trades in equilibrium at its contractual value (its equilibrium value in the absence of default risk). If the State pursues a non-Ricardian FFMP, the IBC of the State (holding with equality) can be treated as a (counterfactual) sovereign bond pricing equation, which determines the *market price* of the bonds. There is, however, no guarantee that the market price of the bonds will equal their contractual value.

The FTPL asserts that, *arbitrary*, non-Ricardian FFMPs - policies for public spending, taxation, interest rates and monetary issuance - need satisfy the IBC of the State *in equilibrium* only, and that they will do so with sovereign debt priced at its contractual value. This surprising result holds because either the price level (in the original FTPL) or the current and anticipated future nominal and real discount factors, driven by the (forward-looking) level of real economic activity (in the New-Keynesian version of the FTPL developed by Sims), adjusts to make the *real* contractual value of the outstanding stock of *nominal* public debt equal to the present discounted value (PDV) of current and future real primary surpluses plus seigniorage.

Necessary conditions for valid dynamic general equilibrium analysis are: (1) the number of equations equals the number of unknowns and the number of state variables equals the number of first-order differential or difference equations and the number of boundary conditions; (2) if condition (1) is satisfied, the resulting system of equations has one or more solutions; and (3) if (1) and (2) are satisfied, the solution(s) make economic sense. Different versions of the FTPL fail at on at least one of these three counts.

The FTPL was developed in the early 1990s by Leeper (1991), Sims (1994), Woodford (1994, 1995, 2001), Cochrane (1998, 2001, 2005) and Bassetto (2002)). It was shown by Buiter (1998, 2001, 2002, 2005) and Niepelt (2004) to be logically inconsistent in all but one class of models, and to be full of extreme, unacceptable anomalies in that one class of models where it is not necessarily logically

[2] For the program and links to the presentations see https://bfi.uchicago.edu/events/next-steps-fiscal-theory-price-level. Other papers attempting to resurrect the FTPL are Sims (2011, 2013, 2016a, c), Leeper (2015) and Cochrane (2016a)).

[3] For a website containing information about this conference, and many of the presentations, see https://bfi.uchicago.edu/events/next-steps-fiscal-theory-price-level. The only (mildly) critical noise was made by Harald Uhlig (2016).

inconsistent: models with a flexible nominal price level, an exogenous rule for the nominal interest rate and a positive stock of nominal government bonds outstanding.

The argument that the FTPL is based on a fallacy and consequently is false was never refuted; its attempted revival is therefore hard to rationalize. One possible reason is that the original FTPL contributions and their refutation did not address the issue of the effective lower bound (ELB) on nominal interest rates. This paper shows that the FTPL remains logically inconsistent and anomaly-ridden at the ELB also. Another possible reason is that the original FTPL was developed for models with a flexible general price level. Sims' recent relaunch of the FTPL in models with nominal price rigidity (which I have dubbed the fiscal theory of the level of economic activity or FTLEA) relies on a different debt re-pricing mechanism from the original FTPL. This paper shows that the FTLEA too is doomed by inconsistencies and anomalies similar to those that invalidate the original FTPL.

The most recent attempts to revive the FTPL are rather bereft of complete formal models. Often only the misspecified government bond pricing equilibrium equation is provided. Only Sims (2011, 2016a) provides explicit Keynesian models - one New-Keynesian (Sims (2011) and one Old-Keynesian (2016a). Both Keynesian models have a predetermined price level, which therefore cannot do the job the FTPL wants it to do. In the New-Keynesian model, with a forward-looking, optimizing consumer, Sims turns the fiscal theory of the price level into a fiscal theory of the level of economic activity (FTLEA), with private consumption supposed to do (indirectly - through the nominal and real discount factors entering the IBC of the State) what the price level tries to do in vain in the original FTPL.

The case against the FTPL is not and never was an empirical one. Nor does it depend on any of its assumptions being viewed as unrealistic. The rejection of the FTPL always rested, and continues to rest, on its logical flaws, inconsistencies and egregious anomalies. An inconsistent theory can have no empirical implications and the realism or lack of it of its assumptions is irrelevant.

1.1. Why it matters

The attempted resurrection of the FTPL matters not just for academic or scholarly reasons. Clearly, propositions and theories that are internally inconsistent must be exposed for what they are and rejected. However, in addition to these academic concerns, there are material real-world policy risks associated with the FTPL/FTLEA: policy disasters could happen if fiscal and monetary policy makers were to become convinced that the FTPL/FTLEA is the appropriate way to consider the interaction of monetary and fiscal policy in driving inflation, aggregate demand, real economic activity and sovereign default risk.

An implication of the FTPL/FTLEA is that monetary and fiscal policy makers – either acting in a cooperative and coordinated manner or acting in an independent and uncoordinated manner – can choose just about any paths or rules for real public spending on goods and services, real taxes net of transfers, interest rates and/or monetary issuance, now and in the future, without having to be concerned about meeting their contractual debt obligations. The general price level (in the classic FTPL) or, real aggregate demand (in the FTLEA version of Sims) is guaranteed to take on the value required to ensure that the real contractual value of the outstanding stock of nominal non-monetary public debt outstanding is always equal to the PDV of the current and future real primary surpluses plus seigniorage of the State.

Because this is nonsense (as shown in Sections 2 and 4) it could be extremely dangerous if taken seriously and acted upon by monetary and fiscal policy makers. After all, what could be more appealing to a politician anxious to curry favor with the electorate through public spending increases and tax cuts, than the reassurance provided by the FTPL, that solvency of the State is never a problem. Explosive sovereign bond trajectories will never threaten sovereign solvency. The general price level or the nominal and real discount factors will do whatever it takes to make the real contractual value of the outstanding stock of nominal government bonds consistent with solvency of the State for arbitrary, non-Ricardian FFMPs. If some misguided government were to take this false theory seriously and were to act upon it, the result, when reality belatedly dawns, could be painful fiscal tightening, government default, excessive recourse to inflationary financing and even hyperinflation.

The risk of the FTPL rubbing off on policy makers could well be real. In a recent note, Katsushiko Aiba and Kiichi Murashima noted - referring to the FTPL as developed by Sims - that *"… the Nikkei and other media have recently reported his prescription for achieving the inflation target based on the FTPL. We should keep a close eye on this theory because PM Abe's economic advisor Koichi Hamada is a believer, meaning that it might be adopted in Japan's future macroeconomic policies"*. (Aiba and Murashima (2017, page 1).

In Brazil, André Lara Resende (2017) argues in a contribution to Valor Econômico, a Brazilian financial newspaper, that high real interest rates in Brazil are simply the result of high nominal interest rates. His analysis is based on the analysis of John Cochrane in Cochrane (2016a), which has the FTPL as one of its key building blocks. If this argument ever gained traction among monetary policy makers in Brazil, it could result in costly policy mistakes.

1.2. What the FTPL is not

1.2a. Unanticipated inflation or financial repression can reduce the real value of nominal public debt

Note that in Standard (non-FTPL/FTLEA) monetary economics too, a change in the general price level changes the real value of the outstanding stock of nominal bonds (private as well as public). Indeed, when faced with imminent default on its debt, a government may well opt for excessive monetary financing of government deficits and for driving up inflation. Unanticipated inflation (that is, inflation that was unanticipated at the time the (fixed rate) nominal debt was issued) can cause the *ex-post*, realized real interest rate on fixed rate nominal debt to be lower that the *ex-ante*, expected real interest rate at the time the debt was issued. Financial repression (keeping nominal interest rates below market rates) can further reinforce the 'unanticipated inflation tax' on holders of nominally denominated government (and private) bonds by turning it into an inflation tax – regardless of whether the inflation is anticipated or unanticipated. This indeed accounts for a sizeable part of the reduction in the general government gross debt-to-GDP ratios after World War II in the UK, the US and many other countries. This has nothing to do with the FTPL – it is Standard economic theory.

In Japan today, the monetary authorities 'target the yield curve' – they set a negative, minus 0.1 percent, interest rate on to the Policy-Rate Balances in current accounts (deposit accounts) held by financial institutions at the Bank of Japan and target the yield on 10-year JGBs at close to zero percent. The 10-year JGBC market is one of the most liquid markets in Japan. It is therefore likely that the expectations hypothesis of the term structure of interest rates would make itself felt at maturities of 10 year. If the 10-year rate is an equilibrium rate, this means that the 10-year rate should be close to the average expected short-term policy rate over the coming ten years, i.e. that the overnight rate is expected to average zero over the next ten years. Term premia (and possibly other risk premia) of course qualify this interpretation. If the markets expected success in the achievement of the 2% per annum inflation target starting, say, 2 years from now, this would mean the average term premium over the next 10 years is expected to be around minus 1.6%. Everything is possible but not everything is likely: term premia in liquid markets cannot be manipulated in such a significant and persistent manner by varying the net supplies that must be absorbed by the non-JGB market participants. Indeed, in Sims (2011) the strict expectations hypothesis of the term structure links the price of a perpetuity (consol) to the instantaneous policy rate.

My interpretation is that the monetary authorities are sending the following message to the fiscal authorities in Japan: "We at the BoJ have done everything we can; we are in a liquidity trap. Further rate cuts won't work and may even be counterproductive by impacting bank profitability negatively. BoJ balance sheet expansion on its own no longer has a material impact on term premia and financial conditions generally. It will only work if the extra fiscal space created by the BoJ balance sheet expansion is used by the fiscal authorities through a fiscal stimulus. So please, fiscal authorities, provide a sizeable additional fiscal stimulus. We will monetize it. And should inflation expectations rise, we promise to use financial repression to cap nominal yields up to 10 years near to zero for a considerable period of time."

What his amounts to is a conventional helicopter money drop combined with financial repression. There is no trace of the FTPL/FTLEA. This issue is analysed further in Section 4.

1.2b. Fiscal dominance, monetary dominance, active or passive monetary and fiscal policy.

The FTPL and non-Ricardian policies are frequently identified with fiscal dominance or active fiscal policy and passive monetary policy. This makes no sense. Ricardian FFMPs can have either monetary or fiscal dominance, as the famous "Unpleasant Monetarist Arithmetic" paper by Sargent and Wallace (1981) shows. Before the public debt reaches the (exogenously given) upper bound, monetary policy is active - the growth rate of the nominal money stock is exogenous. Government borrowing is passive. Once the debt ceiling is reached, monetary growth passively finances the public sector deficit (public spending and taxes don't change). The fiscal dimension of monetary policy (and specifically of central bank monetized balance sheet expansion) exists even if the central bank is operationally independent and even if there is 'monetary dominance' (or active monetary policy and passive fiscal policy) rather than the 'fiscal dominance' (or active fiscal policy and passive monetary policy), that characterizes the 'Unpleasant Monetarist Arithmetic' paper after the ceiling on the government debt-to-GDP ratio is reached. The key insight is that, given the outstanding stocks of State assets and liabilities, if you want to ensure the State remains solvent (if you want a Ricardian FFMP), you cannot specify monetary policy (base money issuance) and fiscal policy (public spending and taxes) independently. Either there is fiscal dominance and monetary issuance becomes endogenously determined (the residual), or there is monetary dominance and public spending and/or taxation have to adjust (becomes the residual) to maintain sovereign solvency.

1.2c. The fiscal theory of seigniorage: monetary policy has an unavoidable fiscal dimension

The size and composition of the balance sheet of the central bank have unavoidable fiscal implications. The fiscal theory of seigniorage (FTS) is the right way of thinking about the inherent fiscal dimension of monetary policy. It is outlined briefly in Section 5 after the FTPL and the FTLEA have been dismantled.

1.3. Outline

The outline of the rest of the paper is as follows. Section 2 develops a flexible price level (New-Classical) model to analyze the FTPL and a sticky price level (New-Keynesian) model to analyze the FTLEA. It uses these models to demonstrate two inconsistencies and five anomalies in the FTPL/FTLEA literature. Section 3 establishes that four anomalies and one inconsistency (possibly two) remain present even when the economy is permanently at the effective lower bound (ELB). Section 4 analyzes an Old-

Keynesian model similar in all important respects to the one developed in Sims (2016a), demonstrates that it has nothing to do with the FTPL/FLTEA, and points out the lack of robustness of its policy prescription. Section 5 shows that the fiscal theory of seigniorage (FTS) is the right way to look at the inherent fiscal dimension of central bank balance sheet expansion and provides the proper analytical foundations for the effectiveness of helicopter money drops.

2. The Fallacy of the Fiscal Theory of the Price Level

2.1. Building blocks for simple formalizations of the FTPL and the FTLEA

I will first state the key results concerning the FTPL and the FTLEA for the case where the economy is never at the ELB. I will choose the sequence of nominal interest rates or the sequence of nominal money stocks in such a way that the instantaneous nominal interest rate on bonds exceeds the nominal interest on money in each period. A continuous time, deterministic model is used. The initial time is $t=0$. We model a closed endowment economy with a public sector (the State, that is, the consolidated Treasury/general government and central bank) and a household sector.

2.1a. The State

I use the following notation: $M(t)\geq 0$ is the nominal stock of central bank money at time *t*; $i^M(t)$ is the instantaneous own interest rate on central bank money; $P(t)$ is the general price level; $\pi(t)\equiv \dot{P}(t)/P(t)$ is the rate of inflation; $B(t)$ is the number of instantaneous or zero-duration nominal bonds issued by the government; the contractual value of that nominal bond is 1 unit of money; $i(t)$ is the instantaneous nominal interest rate on government bonds; $b(t)$ is the number of instantaneous or zero-duration index-linked bonds issued by the government; $r(t)$ is the instantaneous risk-free real interest rate on government debt*;* the nominal contractual value of the index-linked bond at time *t* is $P(t)$; $B^{\ell}(t)$ is the number of nominal consols (perpetuities) paying one unit of money each period (instant) forever; $P^{\ell}(t)$ is the nominal contractual value of a consol; $g(t)$ is real government spending on goods and services (exhaustive public spending); $\tau(t)$ is real taxes net of transfers; s is he real value of the primary surplus of the State; and σ_1 is the nominal value of seigniorage - the change in the nominal base money stock minus any interest paid on the stock of central bank money outstanding.

$$s(t)\equiv \tau(t)-g(t) \qquad (1)$$

$$\sigma_1(t)\equiv \dot{M}(t)-i^M(t)M(t) \qquad (2)$$

There is a second measure of the flow of profits the central bank (and thus the State) derives from the issuance of base money: the difference between the interest rate on non-monetary financial instruments and the interest rate on the monetary base times the outstanding stock of base money. This is denoted σ_2:

$$\sigma_2(t)\equiv \left(i(t)-i^M(t)\right)M(t) \qquad (3)$$

The two flow measures of seigniorage are linked by the 'seigniorage identity', that the PDV of current and future money issuance net-of-interest paid-on-money equals the PDV of current and future interest saved through the issuance of fiat base money rather than non-monetary liabilities, plus the PDV of the terminal value of the stock of fiat base money, minus the value of the initial stock of base money.[4]

$$\int_t^{\infty} e^{-\int_t^v i(u)du}\sigma_1(v)dv \equiv \int_t^{\infty} e^{-\int_t^v i(u)du}\sigma_2(v)dv + \lim_{v\to\infty} e^{-\int_t^v i(u)du}M(v) - M(t) \tag{4}$$

or, equivalently

$$\int_t^{\infty} e^{-\int_t^v r(u)du}\left(\frac{\sigma_1(v)}{P(v)}\right)dv \equiv \int_t^{\infty} e^{-\int_t^v r(u)du}\frac{\sigma_2(v)}{P(v)}dv + \lim_{v\to\infty} e^{-\int_t^v r(u)du}\frac{M(v)}{P(v)} - \frac{M(t)}{P(t)} \tag{5}$$

The instantaneous budget identity of the State (with all bonds priced at their contractual values) is:

$$\frac{\dot{B}(t)}{P(t)} + \frac{P^{\ell}(t)}{P(t)}\dot{B}^{\ell}(t) + \dot{b}(t) \equiv -\left(s(t) + \frac{\sigma_1(t)}{P(t)}\right) + i(t)\frac{B(t)}{P(t)} + \frac{B^{\ell}(t)}{P(t)} + r(t)b(t) \tag{6}$$

We refer to $s(t) + \frac{\sigma_1(t)}{P(t)}$ as the real value of the *augmented* primary surplus of the State. The arbitrage relationships between the rates of return on short nominal bonds, nominal consols and short index-linked bonds are:

$$r(t) = i(t) - \pi(t) \tag{7}$$

and

$$\frac{1}{P^{\ell}(t)} + \frac{\dot{P}^{\ell}(t)}{P^{\ell}(t)} = i(t) \tag{8}$$

The contractual value of the consol is the fundamental or bubble-free forward solution to equation (8), which equals the PDV of all future coupon payments, discounted at (default-) risk-free interest rates

$$P^{\ell}(t) = \int_t^{\infty} e^{-\int_t^v i(u)du}dv \tag{9}$$

Let l denote the real value of the net non-monetary debt of the State:

$$l(t) \equiv \frac{B(t) + P^{\ell}(t)B^{\ell}(t)}{P(t)} + b(t) \tag{10}$$

The intertemporal budget identity of the State is:

$$l(t) \equiv \int_t^{\infty} e^{-\int_t^v r(u)du}\left(s(v) + \frac{\sigma_1(v)}{P(v)}\right)dv + \lim_{v\to\infty} e^{-\int_t^v r(u)du}l(v) \tag{11}$$

Using equation (5), the intertemporal budget identity of the State can also be written as:

$$\frac{M(t)}{P(t)} + l(t) \equiv \int_t^{\infty} e^{-\int_t^v r(u)du}\left(s(v) + \frac{\sigma_2(v)}{P(v)}\right)dv + \lim_{v\to\infty} e^{-\int_t^v r(u)du}\left(\frac{M(v)}{P(v)} + l(v)\right) \tag{12}$$

The solvency constraint of the State is the assumption that the PDV of the terminal value of its *non-monetary* liabilities is non-positive in the limit as the terminal date goes to infinity:

$$\lim_{v\to\infty} e^{-\int_t^v r(u)du}l(v) \le 0 \tag{13}$$

[4] This identity can be established through integration by parts.

Equations (11) and (13) imply the intertemporal budget constraint of the State:

$$l(t) \equiv \frac{B(t)+P^{\ell}(t)B^{\ell}(t)}{P(t)}+b(t) \le \int_t^{\infty} e^{-\int_t^v r(u)du}\left(s(v)+\frac{\sigma_1(v)}{P(v)}\right)dv \tag{14}$$

Equivalently, using equations (12) and (13), the IBC of the State can be written as:

$$\frac{M(t)}{P(t)}+l(t) \le \int_t^{\infty} e^{-\int_t^v r(u)du}\left(s(v)+\frac{\sigma_2(v)}{P(v)}\right)dv+\lim_{v\to\infty} e^{-\int_t^v r(u)du}\frac{M(v)}{P(v)} \tag{15}$$

The fact that the State is not concerned about the PDV of its terminal monetary liabilities, that is, that the solvency constraint of the State is $\lim_{v\to\infty} e^{-\int_t^v r(u)du} l(v) \le 0$ rather than $\lim_{v\to\infty} e^{-\int_t^v r(u)du}\left(\frac{M(v)}{P(v)}+l(v)\right) \le 0$, is a reflection of the assumption that central bank money is *irredeemable*. The holder of X amount of central bank money can never compel the issuer (the central bank) to exchange it for anything other than X amount of central bank money. Central bank money is therefore not in any meaningful sense a liability of the central bank. Central bank money is, however perceived as an asset by the private holders. The household solvency constraint we introduce below is $\lim_{v\to\infty} e^{-\int_t^v r(u)du}\left(\frac{M(v)}{P(v)}+l(v)\right) \ge 0$. This asymmetry between the solvency constraints of the State and of the households is a formalization of the view that central bank money is an asset to the holder but not a liability to the issuer.

The irredeemability of central bank money matters for my analysis of monetary policy effectiveness at the effective lower bound (ELB) in Section 5 below. It is irrelevant for the discussion of the merits of the FTPL/FTLEA, which remains a fallacy even if we replace equation (13) by $\lim_{v\to\infty} e^{-\int_t^v r(u)du}\left(\frac{M(v)}{P(v)}+l(v)\right) \le 0$.

In equation (14) all three bonds are valued at their contractual values. At time *t*, the quantities of the three bonds, $B(t)$, $B^{\ell}(t)$ and $b(t)$, are inherited from the past and are therefore predetermined or given. The IBC imposes on the State the constraint that, for the bonds to be valued at their contractual values, whatever paths or rules it chooses for current and future primary surpluses, seigniorage and policy rates, the PDV of current and future augmented primary surpluses must be *at least* equal to the contractual value of the government bonds outstanding, when the discount rates used in the PDV calculations, including the calculation of the contractual value of the consols, are free of default risk. Paths of or rules for augmented primary surpluses (or for their constituent components (taxes, public spending, seigniorage) and for the policy rate(s) that satisfy equation (14) identically are *Ricardian* FFMPs. The model used here is deterministic; default risk and any other form of risk or uncertainty are not considered. The discount rates are therefore the risk-free nominal and real interest rates.

Of course, governments can pursue and have pursued fiscal-financial-monetary programs that don't always satisfy the IBC. In that case, equation (1.7) can provides a counterfactual test of whether, for a given 'non-Ricardian' FFMP, equation (14) will be satisfied in equilibrium. If it is not, the modeler has to go back to the drawing board. Sovereign insolvency, default and default risk have to be brought into the analysis explicitly. The terms on which the sovereign can access the bond markets when it has defaulted on its debt or is deemed to have a positive probability of defaulting on its debt have to be an integral part of the model. One cannot proceed with the assumption, appropriate with Ricardian FFMPs, that the discount rates applied to future sovereign debt service are free of default risk if the market value of the debt discounted at default-risk-free rates (the RHS of equation (14) or (15)) turns out to be less than the contractual value of the debt (the LHS of equation (14) or (15)).

The IBC of the State, given in equation (13), is a 'primitive' requirement imposed on the State – in and out of equilibrium – if its debt is always priced at its contractual value . It is not just an equilibrium implication of anything the household sector does.

A properly specified equilibrium bond pricing equation version of the IBC of the State, holding with equality, states that the *market value* or *effective value* of the net bond debt of the State equals the PDV of the current and anticipated future primary surpluses of the State (whatever these are), plus the PDV of current and anticipated future seigniorage (whatever that may be), discounted using default-risk-free discount factors. Alternatively (and equivalently), the market value of the bond is the PDV of the future *contractual* payments, discounted using discount factors that incorporate the likelihood that these contractual payments will actually be made and the likelihood that payments less than the contractual payments will be made now and in the future. In the equilibrium bond pricing approach, the FFMP is whatever it is. Essentially arbitrary sequences or rules for public spending, taxes, monetary issuance and policy rates are permitted. Such *overdetermined* FFMPs that are not required to satisfy the IBC of the State (with add debt priced at its contractual value) identically are called *non-Ricardian* FFMPs.

There is no reason to expect that a non-Ricardian FFMP will support a market value of the outstanding net government bond debt that is equal to its contractual value. The PDV of current and future augmented primary surpluses generated by a non-Ricardian FFMP could equal or *exceed* the contractual value of the outstanding net stock of government bonds. If it were to exceed the contractual value, the market value of the bonds equals the contractual value of the bonds. The State is wasting 'fiscal space'.

It is also possible that the PDV of current and future augmented primary surpluses generated by a non-Ricardian FFMP is less than the contractual value of the outstanding net stock of government bonds. In that case, the market value of these bonds is less than their contractual value. In the simple formal models that have been used to analyze the FTPL, the government is in default or insolvent immediately. In more realistic and complex models the government could merely be viewed as having a positive probability of default (that is, of not being able to meet its contractual obligations) at some time in the future.[5]

When the equilibrium bond pricing equation generates a market value for the bonds that is below their contractual value, the market price represents a discount on the contractual price. In Buiter (2001, 2002) I referred to the ratio of the market value of a government bond to its contractual value as the *bond*

[5] In Niepelt (2004), the author argues that even if one accepts the valuation equation approach, this implies an intertemporal budget constraint if one imposes rational expectations and if one goes back to a truly initial period where debt is issued. Once this is accepted, the nominal anchor disappears and the possibility to run "arbitrary" (non-Ricardian) fiscal policies disappears as well—the price level cannot be relied upon to satisfy the IBC. If a bond revaluation factor less than 1 were to occur in the initial period when a government bond is issued, it would not be possible to price that bond at par. The State either sells it at the appropriate discount to its contractual value or the State cannot sell the bond. Niepelt's analysis and mine are substantially the same. I am indebted to Dirk Niepelt for pointing this out to me. Of course, as pointed out in Daniel (2007), if, in that initial period, all the necessary conditions for the FTPL to generate an equilibrium price level sequence are satisfied (flexible prices, exogenous nominal interest rate, non-zero stock of government bonds), the FTPL might be able to pick a price level sequence that yields a unique, non-explosive equilibrium. This amounts to 'relocating' the FTPL to the initial period. Even if there is no inconsistency (overdeterminacy) in this case, the five anomalies introduced below still invalidate the FTPL.

revaluation factor. The bond revaluation factor cannot be greater than 1 or less than 0 (private creditors of the government cannot be turned into private debtors). I denote the bond revaluation factor by $D(t)$.

If we only consider Ricardian FFMPs, the bond revaluation factor will, by construction, always be equal to 1 and can be ignored.

The IBC of the State viewed as a counterfactual government bond pricing equation is given in equation (16) which differs from equation (14) in three ways. First, the weak inequality in equation (14) becomes a strict equality; second, the bond revaluation factor, $D(t)$, in equation (16) transforms the contractual values of the bonds into (counterfactual) effective or market values;[6] third, to emphasize the fact that we are dealing with arbitrary, non-Ricardian FFMPs, I put tildes over the primary surpluses and seigniorage - they are effectively arbitrary and will only by happenstance satisfy equation (16) with $D(t) \geq 1$.

$$D(t)l(t) \equiv D(t)\left(\frac{B(t) + B^{\ell}(t)\int_t^{\infty} e^{-\int_t^v i(u)du} dv}{P(t)} + b(t) \right) = \int_t^{\infty} e^{-\int_t^v r(u)du} \left(\tilde{s}(v) + \frac{\tilde{\sigma}_1(v)}{P(v)} \right) dv \qquad (16)$$

I restrict the analysis to the case where $l(t) > 0$ - the State is a net bond debtor. If it were a net creditor, we would have to verify the solvency of those who issued the bonds held by the State. If $D(t) \geq 1$ the State is solvent (if $D(t) > 1$ the State is 'super-solvent' and is wasting 'fiscal space'). If $D(t) < 1$ (including the case where $D(t) < 0$), the State is insolvent and the assumption maintained in the State's budget identity and solvency constraint, that all sovereign bonds are priced at their contractual values, is falsified. The model is invalid.

In more general models this 'counterfactual' or 'proof by contradiction' checking of the solvency of the State can allow for uncertainty, including price risk driven by any factors other than sovereign insolvency risk, as long it is possible to model the appropriate stochastic discount factors for future augmented primary surpluses and for the calculation of the contractual (default risk-free) value of government bonds in the absence of sovereign default risk.

The FTPL considers non-Ricardian FFMPs but does not add a bond revaluation factor to the equilibrium bond pricing equation. It effectively sets $D(t) \equiv 1$ in the counterfactual bond pricing equilibrium condition (16), which is replaced by:

$$l(t) \equiv \frac{B(t) + B^{\ell}(t)\int_t^{\infty} e^{-\int_t^v i(u)du} dv}{P(t)} + b(t) = \int_t^{\infty} e^{-\int_t^v r(u)du} \left(\tilde{s}(v) + \frac{\tilde{\sigma}_1(v)}{P(v)} \right) dv \qquad (17)$$

Since the FTPL adds an additional equation (the bond pricing equilibrium condition) but does not add another unknown, a model of the economy that has a determinate equilibrium under Ricardian FFMPs should be overdetermined under a non-Ricardian FFMP, that is, mathematically inconsistent with more equations than unknowns.

2.1b. The household sector

[6] I assume that the two nominal bonds and the index-linked bond have the same revaluation factor. This can easily be generalized to allow different revaluation factors for the three types of bonds.

For simplicity, I consider a representative household model. Households are price takers in the markets in which they transact. The household receives an exogenous endowment of a perishable commodity $Y(t)>0$ each period, consumes $C(t)\geq 0$ and pays net real lump-sum taxes $\tau(t)$.

The instantaneous budget identity for the representative household is:

$$\dot{M}+\dot{B}+P^{\ell}\dot{B}^{\ell}+P\dot{b}\equiv P\left(Y-\tau-C\right)+i^{M}M+iB+B^{\ell}+Prb \qquad (18)$$

Equation (18) implies the following intertemporal budget identity for the household:

$$\frac{M(t)}{P(t)}+l(t)\equiv\int_{t}^{\infty}e^{-\int_{t}^{v}r(u)du}\left(C(v)+\frac{\sigma_{2}(v)}{P(v)}+\tau(v)-Y(v)\right)dv+\lim_{v\to\infty}e^{-\int_{t}^{v}r(u)du}\left(\frac{M(v)}{P(v)}+l(v)\right) \qquad (19)$$

The intertemporal household budget identity can also be writes as follows (using equation (5)):

$$l(t)\equiv\int_{t}^{\infty}e^{-\int_{t}^{v}r(u)d}\left(C(v)+\frac{\sigma_{1}(v)}{P(v)}+\tau(t)-Y(v)\right)dv+\lim_{v\to\infty}e^{-\int_{t}^{v}r(u)d}l(v) \qquad (20)$$

The household solvency constraint is the condition that the PDV of its terminal net financial debt cannot be positive:

$$\frac{1}{P(t)}\lim_{v\to\infty}e^{-\int_{t}^{v}i(u)du}\left(M(v)+P(v)l(v)\right)=\lim_{v\to\infty}e^{-\int_{t}^{v}r(u)du}\left(\frac{M(v)}{P(v)}+l(v)\right)\geq 0 \qquad (21)$$

The household's IBC is obtained from equations (19) and (21) or, equivalently, from equations (20) and (21):

$$\frac{M(t)}{P(t)}+l(t)\geq\int_{t}^{\infty}e^{-\int_{t}^{v}r(u)du}\left(C(v)+\frac{\sigma_{2}(v)}{P(v)}+\tau(v)-Y(v)\right)dv \qquad (22)$$

or,

$$l(t)\geq\int_{t}^{\infty}e^{-\int_{t}^{v}r(u)du}\left(C(v)+\frac{\sigma_{1}(v)}{P(v)}+\tau(t)-Y(v)\right)dv-\lim_{v\to\infty}e^{-\int_{t}^{v}r(u)du}\frac{M(v)}{P(v)} \qquad (23)$$

The household IBC will hold with equality if there is non-satiation in consumption and/or real money balances. The household utility function introduced below has that property.

Note for future reference that to the household central bank money definitely is an asset. It can be used to help meet the household solvency constraint given in equation (21).

I ensure that, away from the ELB, central bank money is willingly held by private agents despite being pecuniary-rate-of-return dominated by other financial assets (bonds), by making real money balances an argument in the direct utility function. Making money an argument in a transactions function or production function where it saves on real resources would produce a similar demand for money function.

The representative consumer maximises the utility functional given in equation (24) defined over non-negative sequences of consumption and real money balances subject to (22) or (23) and the initial financial asset stocks given in (25). It takes its endowment income. taxes, prices and interest rates as given. The constant pure rate of time preference is δ.

$$\begin{aligned}&u(t)=\int_{t}^{\infty}e^{-\delta(v-t)}\left[\ln C(v)+\phi\ln\left(\frac{M(v)}{P(v)}\right)\right]dv\\&C(v),M(v)\geq 0;\ \delta>0,\ \phi\geq 0\end{aligned} \qquad (24)$$

$$\begin{aligned} &B(0) = B_0 \\ &B^{\ell}(0) = B_0^{\ell} \\ &b(0) = b_0 \\ &M(0) = M_0 > 0 \end{aligned} \tag{25}$$

Since utility is increasing in consumption and real money balances, the IBC of the household will hold with equality.

The household optimal consumption and money holdings program is characterised by equations (26) and (27), if $i(t) \ge i^M(t)$:

$$\dot{C}(t) = \left(r(t) - \delta\right)C(t) \tag{26}$$

$$\frac{M(t)}{P(t)} = \left(\frac{\phi}{i(t) - i^M(t)}\right)C(t) \tag{27}$$

$$\frac{M(t)}{P(t)} + l(t) = \int_t^{\infty} e^{-\int_t^v r(u)du}\left(C(v) + \frac{\sigma_2(v)}{P(v)} + \tau(v) - Y(v)\right)dv \tag{28}$$

The IBC of the household, holding with equality, is the boundary condition for the optimal consumption program derived from the transversality condition for consumption and the non-satiation property of the utility function. Equations (26), (27) and (28) imply the following 'consumption function':

$$C(t) = \frac{\delta}{1+\phi}\left(\frac{M(t)}{P(t)} + l(t) + \int_t^{\infty} e^{-\int_t^v r(u)du}\left(Y(v) - \tau(v)\right)dv\right) \tag{29}$$

$$i(t) > i^M(t) \ \ , t \ge 0 \tag{30}$$

2.2. The FTPL in the flexible price level model

We first consider the case of a freely flexible general price level. If the endowment is interpreted as labor time, there is continuous full employment in this version of the model: aggregate demand always equals the exogenous endowment. For simplicity, I assume that the perishable 'full employment' endowment, real government spending on the commodity and the nominal interest rate on money are constant in every period:

$$\begin{aligned} &C(t) + g(t) = Y(t) \\ &Y(t) = \overline{Y} > 0 \\ &0 \le g(t) = \overline{g} < \overline{Y} \\ &i^M(t) = \overline{i}^{\,M} \end{aligned} \tag{31}$$

The following conditions will have to be satisfied in any equilibrium where the economy is never at the ELB: the initial conditions for the four financial asset stocks in equation (25) and, for $t \ge 0$:

$$C(t) = \overline{Y} - \overline{g} \tag{32}$$

$$r(t) = \delta \tag{33}$$

$$i(t) = \delta + \pi(t) \tag{34}$$

$$P^{\ell}(t) = \frac{1}{\delta + \overline{\mu}} \tag{35}$$

$$\frac{M(t)}{P(t)} = (\bar{Y} - \bar{g})\left(\frac{\phi}{i(t) - \bar{i}^{M}}\right) \qquad (36)$$

The FTPL adds the IBC of the State, holding with equality and pricing sovereign bonds at their contractual values as an equilibrium condition:

$$l(t) \equiv \frac{B(t) + P^{\ell}(t)B^{\ell}(t)}{P(t)} + b(t) = \int_{t}^{\infty} e^{-\int_{t}^{v} r(u)du} \left(\tilde{s}(v) + \frac{\tilde{\sigma}_1(v)}{P(v)}\right) dv \qquad (37)$$

The Standard (non-FTPL/FTLEA) approach adds the IBC of the government, holding with equality, as a counterfactual sovereign bond pricing equilibrium condition

$$D(t)l(t) \equiv D(t)\left(\frac{B(t) + P^{\ell}(t)B^{\ell}(t)}{P(t)} + b(t)\right) = \int_{t}^{\infty} e^{-\int_{t}^{v} r(u)du} \left(s(v) + \frac{\sigma_1(v)}{P(v)}\right) dv \qquad (38)$$

When Ricardian FFMPs are considered, $D(t) \equiv 1$ and equation (38) can be omitted from the rest of the analysis, which can proceed because the maintained assumption in the model - that sovereign bonds trade at their contractual values – has been verified. When non-Ricardian FFMPs are considered, the model is solved including the counterfactual sovereign bond pricing equation (38). If, in equilibrium, $D(t) \geq 1,\ t \geq 0$, the maintained assumption in the model - that sovereign bonds trade at their contractual values – has been verified and the analysis can proceed. If $D(t) < 1$, the maintained assumption is falsified and the model is inconsistent.

Note that, if we substitute the goods market equilibrium condition (32) into the household's IBC (holding with equality) (28), we obtain the something very close to the IBC of the State, holding with equality:

$$\frac{M(t)}{P(t)} + l(t) = \int_{t}^{\infty} e^{-\int_{t}^{v} r(u)du} \left(s(v) + \frac{\sigma_2(v)}{P(v)}\right) dv \qquad (39)$$

or, equivalently

$$l(t) = \int_{t}^{\infty} e^{-\int_{t}^{v} r(u)d} \left(s(v) + \frac{\sigma_1(v)}{P(v)}\right) dv - \lim_{v \to \infty} e^{-\int_{t}^{v} r(u)d} \frac{M(v)}{P(v)} \qquad (40)$$

Equation (39) differs from the IBC of the State (holding with equality) given in equation (15) and equation (40) differs from the IBC of the State (holding with equality) given in equation (14) by the term $\lim_{v \to \infty} e^{-\int_{t}^{v} r(u)d} \frac{M(v)}{P(v)}$, which is subtracted from the RHS of equations (15) and (14) to get (39) and (40). As noted earlier, this term occurs because of the assumption that central bank money is an irredeemable 'liability' of the State and plays no role in the demonstration that the FTPL/FTLEA is a fallacy.

We are now ready to consider the inconsistencies and anomalies in the FTPL.

2.2a. The nominal money stock as the policy instrument

With A Friedmanian central bank that implements monetary policy by setting rules for the nominal money stock, we immediately have an inconsistency:

Inconsistency 1: ***When the nominal money stock is the policy instrument, the FTPL results in an overdetermined model.***

Consider the case where the central bank sets the nominal stock of money either through some exogenous open-loop rule or through some feedback rule that does not depend on past, present and anticipated future values of nominal interest rates. The instantaneous nominal interest rate on bonds, $i(t)$, is endogenously determined in this case.

Until the discussion of the economy at the ELB in Section 3, I consider only monetary rules that ensure that the (endogenous) nominal interest rate on bonds is always above the nominal interest rate on money, so the economy is never at the ELB. The following very simple rule with a constant proportional growth rate, $\bar{\mu}$, for the nominal money stock has that property:

$$\begin{aligned} &\frac{\dot{M}}{M} = \bar{\mu} > i^M - \delta \\ &M(0) = M_0 > 0 \end{aligned} \tag{41}$$

I will also consider a simple non-Ricardian policy rule for public spending, taxes and seigniorage: the real value of the 'augmented primary surplus' of the State, $z \equiv s + \frac{\sigma_1}{P}$ is constant. Given the earlier assumptions about public spending and monetary growth, this means that

$$\tau(t) = \bar{z} + \bar{g} - \left(\bar{\mu} - i^M\right)\frac{M(t)}{P(t)} \tag{42}$$

Finally, as there are three different types of bonds, I have to make assumptions about the paths followed by the quantities outstanding of these three bonds. I assume that the number of short nominal bond issues and the number of consol issues follow exogenous sequences:

$$\begin{aligned} &\dot{B}(t) = \dot{\bar{B}}(t) \\ &\dot{B}^{\ell}(t) = \dot{\bar{B}}^{\ell}(t) \\ &B(0) = \bar{B}_0 \\ &B^{\ell}(0) = \bar{B}_0^{\ell} \\ &b(0) = \bar{b}_0 \end{aligned} \tag{43}$$

The issuance of index-linked bonds $\dot{b}(t)$ then takes whatever value is required to satisfy the budget and balance sheet identities of the State:

$$\dot{b}(t) \equiv -\bar{z} + (\delta + \bar{\mu})\frac{\bar{B}(t)}{P(t)} + \frac{\bar{B}^{\ell}(t)}{P(t)} + \delta b(t) - \frac{\dot{\bar{B}}(t)}{P(t)} - \left(\frac{1}{\delta + \bar{\mu}}\right)\frac{\dot{\bar{B}}^{\ell}(t)}{P(t)} \tag{44}$$

This results in the IBC of the State, holding with equality and with all bonds priced at their contractual values, simplifying to:

$$\frac{1}{P(t)}\left(\bar{B}(t) + \frac{\bar{B}^{\ell}(t)}{\delta + \bar{\mu}}\right) + b(t) = \frac{\bar{z}}{\delta} \tag{45}$$

It is well-known that, with flexible prices, there are infinitely many equilibria under this policy rule. There is the barter equilibrium where money has no value at any point in time $\frac{1}{P(t)} = 0,\ \ t \geq 0$. Then there are infinitely many non-fundamental, bubble or sunspot equilibria where, despite the fact that all the exogenous variables of the model (including the growth rate of the nominal money stock) are constant, the rate of inflation is non-stationary. I will not consider the barter equilibrium or the inflationary or deflationary bubbles (see Buiter and Sibert (2007)). I will only consider the unique stationary equilibrium in which the inflation rate is constant. In that equilibrium, the following conditions hold:

$$\pi(t) = \bar{\mu} \tag{46}$$

$$i(t) = \bar{\mu} + \delta > \bar{i}^M \tag{47}$$

The inflation rate equals the constant growth rate of the nominal money stock. The nominal interest rate on bonds exceeds the interest rate on money. The sequences of the nominal money stock and the general price level are obtained from equation (41) and

$$\frac{M_t}{P_t} = (\bar{Y} - \bar{g})\left(\frac{\phi}{\bar{\mu} + \delta - \bar{i}^M)}\right) \tag{48}$$

The monetary equilibrium condition (48) tells us that the real money stock is constant and that the price level sequence increases at the same proportional rate as the nominal money stock. So, the price level sequence is determined without bringing in the IBC of the State, holding with equality and with sovereign bonds priced at their contractual values. If we insist on treating equation (45) as an equilibrium condition we have an overdetermined system, with the price level determined twice, once by the monetary equilibrium condition (48) and the current value of the nominal money stock (which is determined by the initial value of the nominal money stock and the constant growth rate of the nominal stock from the initial date), and once more by the IBC of the State. With $\bar{B}(t), \bar{B}^\ell(t)$ and $b(t)$ predetermined and $\bar{z}$ exogenous (and constant in this particular example) and with $P(t)$ determined by the monetary equilibrium condition, the IBC of the State will be satisfied only by happenstance.

The Standard approach would replace equation (45) by:

$$D(t)\left[\frac{1}{P(t)}\left(\bar{B}(t) + \frac{\bar{B}^\ell(t)}{\delta + \bar{\mu}}\right) + b(t)\right] = \frac{\bar{z}}{\delta} \tag{49}$$

If equation (49) solves for a value of $D(t)$ greater than or equal than 1, the analysis can proceed. If it solves for a value of $D(t)$ less than 1, it is back to the drawing board, because the maintained hypothesis, that sovereign debt is priced at its contractual value, is falsified.

2.2b The nominal interest rate as the monetary policy instrument

I shall only consider exogenous (time-contingent or open-loop) rules for the nominal interest rate. All results go through for feedback rules/closed-loop rules/contingent rules as long as the nominal interest rate is not made a function of current, past or anticipated future values of nominal variables like the nominal money stock or the nominal price level. For simplicity, I assume that the instantaneous nominal interest rate on bonds is constant:

$$i(t) = \bar{i} > \bar{i}^M \tag{50}$$

This implies:

$$P^\ell(t) = \frac{1}{\bar{i}} \tag{51}$$

When the nominal interest rate is the policy instrument, the nominal stock of money is endogenous. The equilibrium under a constant nominal interest rate rule simplifies to equations (32), (33), (25) and:

$$\pi(t) = \bar{i} - \delta \tag{52}$$

$$\frac{M(t)}{P(t)} = (\overline{Y} - \overline{g})\left(\frac{\phi}{\overline{i} - \overline{i}^{M}}\right) \tag{53}$$

$$\frac{1}{P(t)}\left(\overline{B}(t) + \frac{\overline{B}^{\ell}(t)}{\overline{i}}\right) + b(t) = \frac{\overline{z}}{\delta} \tag{54}$$

Note that equation (52) determines a constant rate of inflation equal to the difference between the (constant) policy-determined nominal interest rate on bonds and the constant real interest rate (equal to the constant rate of time preference, δ). Equation (53) determines a constant stock of real money balances for every period.

Without the IBC of the State in equation (54), treated as an equilibrium condition, the nominal policy rate-pegging model has an indeterminate general price level and nominal money stock. The real stock of money balances is uniquely determined, but with the nominal money stock endogenous, there is no nominal anchor. The nominal interest rate on bonds is not a nominal anchor: $i(t) - \overline{i}^{M}$ is the real rate of return differential between nominal short bonds and money.

It might seem that the IBC of the State in equation (54) can come to the rescue here by providing another equilibrium condition to pin down the general price level. This turns out to be an illusion. Allowing equation (54) as an additional equilibrium condition results in a number of egregious anomalies.

Anomaly 1: ***Negative price levels or, an arbitrarily restricted domain of existence even under a nominal interest rate rule***

Consider the equilibrium conditions for the exogenous interest rate rule under the non-Ricardian FFMP, given by equations (51), (52), (53), (54) and (44). I rewrite the IBC of the State as follows:

$$\frac{\overline{B}(t) + \left(\overline{B}^{\ell}(t) / \overline{i}\right)}{P(t)} = \frac{\overline{z}}{\delta} - b(t)$$

A negative general price level is not considered a desirable property of a monetary equilibrium model. Therefore, if there is a positive net stock of nominal public debt ($B(t) + \left(B^{\ell}(t) / \overline{i}\right) > 0$), we require, in order to ensure $P(t) > 0$, that the PDV of current and future augmented primary State budget surpluses exceeds the value of the outstanding stock of index-linked debt. If the initial net stock of nominal government debt is negative (the State is a creditor as regards nominal bond instruments) the PDV of current and future primary State budget surpluses plus the PDV of current and future seigniorage minus the value of the outstanding stock of index-linked debt has to be negative. Both cases seem to represent arbitrary restrictions on the feasible domain of non-Ricardian FFMPs and on the split between nominal bond financing and index-linked bond financing.

Anomaly 2: ***Even under a nominal interest rate rule, the FTPL breaks down if there is no net nominal government bond debt outstanding***

If there are only index-linked State bonds, there are no nominal bonds that can be priced to satisfy the IBC of the State: $b(t) = \frac{\overline{z}}{\delta}$ will only be satisfied by happenstance. The same would be the case in an open economy if all the bond debt of the sovereign is denominated in foreign currency. With only index-linked debt, the price level and the nominal money stock are indeterminate, and the IBC of the State will in general be violated, so the model is also inconsistent.

The counterfactual test for government solvency works just as well in a world with only index-linked debt. If the equation $D(t)b(t)=\frac{\overline{z}}{\delta}$ solves for $D(t)\geq 1$ all is well and the analysis can proceed. If $D(t)<1$, the maintained hypothesis that sovereign bonds are priced at their contractual values is refuted. The model is not fit for purpose.

Anomaly 3: ***Under a nominal interest rate rule, the FTPL can determine the price of money in a world without money - it can price phlogiston and determine an equilibrium price without a quantity***

The FTPL can determine the price of money (the reciprocal of the general price level) even if money does not exist except as an imaginary, abstract numeraire or unit of account. Consider the special case of our model where there is no monetary asset: there is no demand for or supply of money. Money does not exist as a store of value, means of payment, transactions medium or medium of exchange. It has no existence as a physical currency, with or without intrinsic value, as a book-keeping entry or as e-money. There is something called money that, for some reason, serves as the unit of account, numéraire or invoicing unit. A government bond is denominated in terms of that abstract, imaginary unit of account. In our model, we achieve this by setting $M(t)\equiv 0$, $t\geq 0$, $\overline{\mu}\equiv 0$, $i^{M}\equiv 0$ and $\phi=0$. The monetary equilibrium condition (equation (53)) is gone, but the IBC of the State (equation (54)) and the single-period budget identity of the State plus the rule determining the composition of bond funding (summarized in equation (44)), still determine the sequence of equilibrium price levels (provided these are not negative).

Instead of something non-existing called 'money', we could use another abstract/imaginary numéraire – phlogiston, say, the substance formerly believed to be embodied in all combustible materials. It this world, when the FTPL supports a positive general price level (see Anomaly 1), it manages to price non-existent phlogiston, just as it can price non-existent money. I consider this to be an undesirable, indeed unacceptable feature of the model.

To illustrate the deep conceptual bizarreness of the phlogiston economy, consider what a one-period maturity pure discount nominal bond actually is in such an economy. It promises, in period *t*, to pay the purchaser, 'something' in period *t+1*. That something cannot be one unit of phlogiston, because phlogiston does not exist except as a unit of account. Instead it promises to pay the holder in period *t+1 something worth one unit of phlogiston* in that period. How do we know what a unit of phlogiston is worth in period *t+1* – in terms of things that actually exist other than as pure numéraires? We have this phlogiston-denominated bond equilibrium pricing condition in every period. It tells us that the real value of the phlogiston-denominated bond, valued at its contractual value in terms of phlogiston, has to be equal to the PDV of the current and future real augmented primary budget surpluses of the State.

So, in a world where money does not exist except as a pure numéraire, a nominal bond is the ultimate *non-deliverable* forward contract.[7] I believe that it makes no sense to model a world where non-deliverable contracts exist without there also being a deliverable benchmark. Money has to exist either as

[7] According to Investopedia *"A non-deliverable forward (NDF) is a cash-settled, short-term forward contract in a thinly traded or nonconvertible foreign currency against a freely traded currency, where the profit or loss at the settlement date is calculated by taking the difference between the agreed upon exchange rate and the spot rate at the time of settlement, for an agreed upon notional amount of funds. The gain or loss is then settled in the freely traded currency"*, http://www.investopedia.com/terms/n/ndf.asp. The key relevant point is that no payment in the thinly traded or nonconvertible currency is ever made. All payments are made in the freely traded currency. The amount of the freely traded currency paid is given by the notional amount of the contract times the difference between the agreed upon forward rate and the spot rate at the time of settlement.

a commodity (with or without intrinsic value) or as a financial claim issued by some economic entity. There has to be a benchmark spot market for money and a deliverable forward contract for money if a non-deliverable forward contract for money is to make sense. In the preceding paragraph the word ‘money’ can be replaced by ‘phlogiston’.

The FTPL fails this test, insofar as it can price money (phlogiston) is a world where there are no deliverable spot or forward contracts for money (phlogiston). I recognize this is an anomaly rather than a logical inconsistency. I do, however, consider this anomaly to be as devastating as the logical inconsistencies inherent in the FTPL: it is inconceivable to me to work with a model of the economy that can determine an equilibrium price of something without an associated quantity of that something.

Anomaly 4: ***Under an exogenous nominal interest rate rule, the FTPL can be replaced by the HTPL***

If the FTPL is deemed appealing, how about the household theory of the price level (HTPL) or even the Mr Jones TPL? Assume the State follows a Ricardian FFMP and therefore always satisfies its IBC. Indeed, assume that this is an optimizing, forward-looking State that does not waste any fiscal resources. Its IBC therefore holds with equality – and so will the IBC of the household, in equilibrium (if an equilibrium exists). Consider the household’s IBC in equation (23), but replace the household’s bond claims on the government with household (non-monetary) debt (which can of course be negative). Bonds issued by the household have a superscript *h*.

Assume that the household follows a non-Ricardian consumption and portfolio allocation rule: it chooses an exogenous constant sequence for its augmented primary surplus:

$$\bar{z}^h = \bar{Y} - \left(C(t) + \tau(t) + \frac{\dot{M}(t) - i^M M(t)}{P(t)} \right) \quad , \quad t \geq 0 \tag{55}$$

Assume that the household also specifies rules for the composition of the bonds it holds.

I will assume that the PDV of the terminal value of the stock of base money, $\lim_{v\to\infty} e^{-\int_t^v r(u)d} \frac{M(v)}{P(v)}$, is zero, as is typically assumed in the FTPL literature. In that case, the IBC of the household looks as follows in equilibrium:

$$\frac{\bar{B}^h(t) + \left(\bar{B}^{\ell,h}(t) / \bar{i}\right)}{P(t)} + b^h(t) = \frac{\bar{z}^h}{\delta} \tag{56}$$

$$\frac{\dot{\bar{B}}^h(t)}{P(t)} + \frac{\dot{\bar{B}}^{\ell,h}(t)}{\bar{i}P(t)} + \dot{b}^h(t) \equiv -\bar{z}^h + \left(\bar{i}\,\frac{\bar{B}^h(t)}{P(t)} + \frac{\bar{B}^{\ell,h}(t)}{P(t)} + \delta b^h(t) \right) \tag{57}$$

The household chooses the mix of bonds in its portfolio with a rule like the one given in equation (43).

The monetary equilibrium condition under the exogenous nominal interest rate rule is the same as under the FTPL. The nominal money stock is endogenous. The household IBC, given in equation (56), determines the general price level (subject to reservations similar to the other Anomalies of the FTPL). This would be the household theory of the price level or HTPL.

Note that the HTPL need not to apply to the IBC of the household sector as a whole. The State could follow a Ricardian FFMP and every household but one could follow “Ricardian“ consumption and asset allocation rules. The one exception, Mr. Jones, say, follows a non-Ricardian consumption and asset allocation rule. Mr Jones’s IBC, holding with equality, would be the equilibrium nominal bond pricing equation determining the general price level.

I hope that anyone who considers the plausibility of the HTPL concludes that it is without merit. And so it is; and it is formally identical to the FTPL.

Anomaly 5: ***When the sovereign bond pricing equilibrium condition is specified correctly, there is no FTPL***

Consider the model under the exogenous interest rate rule, but with the FTPL version of the IBC of the State, holding with equality and with the bonds priced at their contractual values (equation (37)), replaced with equation (38) - the counterfactual equilibrium bond pricing equation. Under the non-Ricardian FFMP of this section, equation (38) becomes:

$$D(t)\left(\frac{\bar{B}(t)+\bar{B}^{\ell}(t)/\bar{i}}{P(t)}+b(t)\right)=\frac{\bar{z}}{\delta} \tag{58}$$

Assume that the conditions for Anomaly 1 (a negative price level) and Anomaly 2 (an overdetermined system because all debt is index-linked) to hold are not satisfied, so the FTPL produces a positive price level from equation (54). A sufficient set of conditions would be $\bar{B}(t)+\bar{B}^{\ell}(t)/\bar{i}>0$, $b(t)>0$ and $\bar{z}>0$. The counterfactual equilibrium bond pricing equation (58) in this case determines the real market value of the outstanding government bonds, $D(t)l(t)=D(t)\left(\frac{\bar{B}(t)}{P(t)}+\frac{\bar{B}^{\ell}(t)}{\bar{i}P(t)}+b(t)\right)$. Neither $P(t)$ nor $D(t)$ are individually determinate however. If there are no net nominal bonds outstanding, $\bar{B}(t)+\bar{B}^{\ell}(t)/\bar{i}=0$, then the debt revaluation factor (a real variable) is uniquely determined from equation (58). The general price level is indeterminate, however.

2.3 The FTLEA in action in models with a sticky general price level

Because the entire thrust of the FTPL is to make the general price level do the work of the bond revaluation factor, it would seem pretty self-evident that in models with a predetermined or sticky general price level (any Old-Keynesian or New-Keynesian model), the FTPL would find itself facing the familiar problem of an overdetermined system, with the general price level determined twice – once by the IBC of the State and once by history. That presumption is not quite correct. Models with a sticky general price level are not necessarily overdetermined (more equations than unknowns). Even if they are not overdetermined, however, they are generically inconsistent.

Inconsistency 2: ***The FTLEA implies either an overdetermined model or a model that is inconsistent in another way when the price level is sticky, even under an interest rate rule***

In Section 2.3 and Section 4, the general price level and the rate of inflation are predetermined and updated through an accelerationist Phillips curve; let $y=\ln Y$ and $\bar{y}=\ln\bar{Y}$:

$$\begin{aligned}\dot{\pi}(t)&=\alpha\left(y(t)-\bar{y}\right)\\ \alpha&>0\end{aligned} \tag{59}$$

Both the inflation rate, $\pi(t)$, and the price level, $P(t)$, are predetermined in this model. Actual output, $Y(t)$, can differ from the exogenous and constant level of potential output, $\bar{Y}$. Actual output is demand-determined:

$$Y(t)=C(t)+\bar{g} \tag{60}$$

We keep the rest of the model the same as before, except for the interest rate rules and the rules governing public spending, taxation and money issuance, and assume that the nominal interest rate is the policy instrument, with the nominal money stock endogenous. The optimizing, forward-looking household whose optimization problem is described in equations (22), (24) and (25) follows a Ricardian consumption and asset accumulation plan and its IBC holds with equality, implying that, in equilibrium (should an equilibrium exist), the IBC of the State also holds with equality.[8] To simplify the exposition further, I will assume $\overline{g} = 0$ in what follows; let $c = \ln C$

The model can be summarized as follows, for $t \geq 0$:

$$\dot{c}(t) = i(t) - \pi(t) - \delta \tag{61}$$

$$\dot{\pi}(t) = \alpha\left(c(t) - \overline{y}\right) \tag{62}$$

$$\frac{\dot{P}(t)}{P(t)} = \pi(t) \tag{63}$$

$$\begin{aligned} &\dot{B}(t) + \dot{B}^{\ell}(t)\int_t^{\infty} e^{-\int_t^v i(u)du} dv + P(t)\dot{b}(t) \equiv \\ &-\left(P(t)s(t) + \sigma_1(t)\right) + i(t)B(t) + B^{\ell}(t) + \left(i(t) - \pi(t)\right)P(t)b(t) \end{aligned} \tag{64}$$

$$\dot{B}^{\ell}(t) = \dot{\overline{B}}^{\ell}(t) \tag{65}$$

$$\dot{b}(t) = \dot{\overline{b}}(t) \tag{66}$$

$$\frac{M(t)}{P(t)} = \left(\frac{\phi}{i(t) - \overline{i}^M}\right)c(t) \tag{67}$$

$$\frac{B(t) + B^{\ell}(t)\int_t^{\infty} e^{-\int_t^v i(u)du} dv}{P(t)} + b(t) = \int_t^{\infty} e^{-\int_t^v (i(u)-\pi(u))du} \left(s(v) + \frac{\sigma_1(v)}{P(v)}\right) dv \tag{68}$$

$$\begin{aligned} \pi(0) &= \pi_0 \\ P(0) &= P_0 \\ B(0) &= B_0 \\ B^{\ell}(0) &= B_0^{\ell} \\ b(0) &= b_0 \end{aligned} \tag{69}$$

For the moment, consider the nominal interest rate, $i(t)$, the real primary surplus, *s(t)* and the real value of seigniorage $\frac{\sigma_1(t)}{P(t)}$ to be exogenously given for all $t \geq 0$.

The model has six first-order differential equations (equations (61) to (66)) and six state variables, π, P, B, B^{ℓ}, b and c; the first five of them are predetermined, with initial conditions given in equation (69).

[8] Until Section 5, I assume that $\lim_{v \to \infty} e^{-\int_t^v r(u)d} \frac{M(v)}{P(v)} = 0$

Consumption is non-predetermined. Its boundary condition is the intertemporal budget constraint of the household, holding with equality or, equivalently (in equilibrium – if an equilibrium exists, that is), the intertemporal budget constraint of the State, holding with equality, given in equation (68). There is one endogenous variable that is not a state variable, the nominal money stock, whose value is determined by the monetary equilibrium condition (67). So, we have the same number of equations and unknowns, the same number of state variables and first-order differential equations and the right number of boundary conditions (both initial conditions and terminal conditions). Does that mean all is well with the FTLEA? Meeting the 'counting tests' is just a necessary condition for the system to have one or more solutions. It means that the system is not overdetermined, but the equations describing it still may not have a solution – may be inconsistent. Finally, even if the equations have one or more solutions, these solutions may not make economic sense (the HTPL and the ability to price phlogiston are two examples).

Consider again the IBC of the State, holding with equality and with government bonds priced at their contractual values:

$$\frac{B(t)+B^{\ell}(t)\int_t^{\infty}e^{-\int_t^v i(u)du}dv}{P(t)}+b(t)=\int_t^{\infty}e^{-\int_t^v r(u)du}\left(\tilde{s}(v)+\frac{\tilde{\sigma}_1(v)}{P(v)}\right)dv \tag{70}$$

It is clear that, with the price level predetermined ($P(t)$ is given), the FTPL cannot hold: the price level cannot take on whatever value is required ensure that equation (70) holds. Sims (2011) argues that the discount factors can ensure that equation (70) holds despite $P(t)$ being given by history. The real discount factors, $e^{-\int_t^v r(u)du}=e^{-\int_t^v (i(u)-\pi(u))du}$, $v\ge t$, that discount current and future non-Ricardian real augmented primary surpluses and, if there is long-dated nominal debt like nominal consols, the nominal discount factors, $e^{-\int_t^v i(u)du}$, $v\ge t$, are, according to Sims, capable of ensuring that the IBC of the State holds. They can take on the required values because real consumption, C, is a non-predetermined state variable when the household is optimizing, forward-looking and able to borrow and lend in efficient capital markets. Current and future values of C will take on the values required to make the current and future discount factors take on the values to ensure that equation (70) holds for all $t\ge 0$. This theory I refer to as the fiscal theory of the level of economic activity or FTLEA.

Simply stating this proposition ought to be enough to convince the reader that it cannot be true for arbitrary non-Ricardian policies. It is trivial to come up with non-Ricardian FFMPs that will cause equation (68), to be violated. Consider the non-Ricardian FFMP in equation (71):

$$\begin{aligned} &i(t)=\delta+\pi(t),\ \delta>0,\ t\ge 0 \\ &s(v)+\frac{\sigma_1(v)}{P(v)}=\bar{z}+\frac{B^{\ell}(t)}{P(t)}e^{-\int_t^v \pi(u)du},\ v\ge t\ge 0 \end{aligned} \tag{71}$$

Under this FFMP equation (68) becomes:

$$\frac{B(t)}{P(t)}+b(t)=\frac{\bar{z}}{\delta} \tag{72}$$

With every variable in equation (72) exogenous or predetermined, it will only hold by happenstance. Next consider the non-Ricardian FFMP in equation (73):

$$i(t) = \text{ anything at all, } t \geq 0$$
$$s(v) + \frac{\sigma_1(v)}{P(v)} < 0, \; v \geq t \qquad (73)$$

Assume that $B(t) \geq 0, B^{\ell}(t) \geq 0$ and $b(t) \geq 0$ with at least one of these three inequalities holding strictly. A permanent augmented primary deficit obviously cannot generate the resources required to service a positive net stock of sovereign debt.

The model just analyzed is a simplified version of the New Keynesian model analyzed in Sims (2011). Sims (2011) is actually a 'phlogiston model' – there is no money in it except as a numeraire and the unit in which the contractual payments due on the nominal bonds are denominated. Setting $M(t) = \dot{M}(t) = \sigma_1(t) = \sigma_2(t) = \phi = 0, \; t \geq 0$ and omitting the monetary equilibrium condition (67), does not alter the conclusion that non-Ricardian FFMPs, including the two just analyzed, can cause the IBC of the State to be violated. The proposition that the nominal and real discount factors in the IBC of the State (equation (68)) can cause it to hold even for arbitrary rules for $s(t) + \frac{\sigma_1(t)}{P(t)}$, the augmented primary surplus, and for the policy rate, $i(t)$, is incorrect. For simplicity (and to maximize comparability with the Sims (2011) New-Keynesian model), I will omit money the in models analyzed in the rest of this section. Since we also assume that real government spending on goods and services is zero, the real value of the augmented primary surplus equals real taxes net of transfers: $s + \frac{\sigma_1}{P} = \tau$.

There no doubt exist non-Ricardian rules that satisfy the IBC of the State in equilibrium - even equation (72) could be satisfied, given the historically determined values of the debt stocks and the price level and the exogenously given pure rate of time preference, if the authorities picked the unique value of $\bar{z}$ for which equation (72) holds.

There is, however, no set of necessary and sufficient conditions that guarantee that non-Ricardian (essentially arbitrary) FFMPs will satisfy the IBC of the State with equality in equilibrium. Neither the price level nor the real and nominal discount factors can be counted on to make non-Ricardian FFMPs consistent with sovereign solvency.

2.3a. A Ricardian and a 'realistic' non-Ricardian FFMP in the New-Keynesian model

It is easy to come up with a Ricardian set of rules for the nominal interest rate and the augmented primary surplus that ensures that the model of equations (61) to (69) is well - behaved. The Ricardian FFMP is given in equations (74) and (75).

$$\begin{aligned} & i(t) = \delta + \pi^* + \beta_1\left(\pi(t) - \pi^*\right) + \beta_2\left(y(t) - \bar{y}\right) \\ & \beta_1 > 1, \; \beta_2 > 0 \end{aligned} \qquad (74)$$

$$\begin{aligned} & s(t) + \frac{\sigma_1(t)}{P(t)} = \tau(t) = r(t)l(t) + \zeta\left(l(t) - l^*\right) \\ & \zeta > 0 \end{aligned} \qquad (75)$$

The interest rate rule is a Taylor rule, where the nominal policy rate increases with the excess of the inflation rate over the target rate of inflation π^* and the output gap. Setting $\beta_1 > 1$ means that the real policy rate rises when the rate of inflation rises; this tends to be stabilizing in a variety of models. The real augmented primary surplus equals the real interest rate bill plus some fraction of the gap between the actual real stock of debt and its target value, l^*. With the Taylor rule and the convergent real public debt rule, the economic system can be reduced to a system of three first-order linear differential equations.

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \\ \dot{l} \end{bmatrix} = \begin{bmatrix} \beta_2 & \beta_1 - 1 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & -\zeta \end{bmatrix} \begin{bmatrix} c \\ \pi \\ l \end{bmatrix} + \begin{bmatrix} (1-\beta_1)\pi^* - \beta_2 \bar{y} \\ -\alpha \bar{y} \\ \zeta l^* \end{bmatrix} \tag{76}$$

Under this Ricardian FFMP, the real stock of government debt converges smoothly to its target value l^*, unaffected by the dynamics of c and π, whose dynamics are in turn unaffected by the dynamics of l. In the unique steady state, $c = \bar{y},\ \pi = \pi^*, l = l^*$ and $r = \delta$.

The two eigenvalues, λ_1 and λ_2 governing the dynamics of consumption and inflation satisfy:

$$\begin{aligned} \lambda_1 + \lambda_2 &= \beta_2 > 0 \\ \lambda_1 \lambda_2 &= \alpha(1-\beta_1) < 0 \end{aligned} \tag{77}$$

So, both roots are real, one root is positive and one negative. With one predetermined state variable (inflation) and one non-predetermined state variable (consumption), there is a unique solution (saddle path) that converges to the unique steady state. The third eigenvalue, which only drives the real debt stock, is $\lambda_3 = -\zeta < 0$ if $\zeta > 0$). The real stock of government debt converges exponentially to its target value l^*.

This is completely standard, non-FTPL and non-FTLEA economics. The FFMP is Ricardian and guarantees that the IBC of the State is always satisfied. It is hard to imagine any fiscal rule robustly satisfying the IBC of the State with equality and with sovereign debt priced at its contractual value, unless there is feedback by the real augmented primary surplus, $s + \frac{\sigma_1}{P} = \tau$, from the level of the real stock of outstanding non-monetary public debt, l, sooner or later, directly or indirectly (in more general models this could be through the two other state variables, c and π). The real interest bill, rl, produces a ‘snowball effect’ in the debt dynamics if the real interest rate is positive (in models with positive growth of potential output, the snowball effect is represented by the term $(r-\gamma)\frac{l}{Y}$ where $\gamma = \dot{Y}/Y$). Our model has a positive real interest rate equal to the pure rate of time preference in steady state. These explosive dynamics have to be neutralized by the augmented primary surplus. It is therefore surprising that in the Sims (2011) New Keynesian model, the numerical simulations support stable, convergent behavior even though there is no feedback by the augmented primary surplus from the debt stock.[9] In terms of the New-Keynesian model analyzed in this paper, the Sims non-Ricardian FFMP can be approximated reasonably well by our Taylor rule (equation (74)) and the following fiscal rule:

$$\begin{aligned} & s + \frac{\sigma_1}{P} = \tau = \omega c \\ & \omega > 0 \end{aligned} \tag{78}$$

Equation (78) is rather like an automatic fiscal stabilizer, where real tax receipts net of transfers vary positively with the level of real economic activity.

[9] Since Sims’s (2011) New-Keynesian model does not include money, the augmented primary surplus is the same as the regular primary surplus. This does not change the analysis.

The dynamics of consumption and inflation are the same as before – linear with the eigenvalues λ_1 and λ_2 given in equation (77). The debt accumulation equation under the fiscal rule in (78) is non-linear, and I take a linear approximation at the unique steady state where $c = \bar{c} = \bar{y}$, $\pi = \bar{\pi} = \pi^*$, $l = \bar{l} = \dfrac{\omega \bar{y}}{\delta}$ and $r = \bar{r} = \delta$.

The dynamics of c and π are again independent of the behavior of the dynamics of l – and vice versa.

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \\ \dot{l} \end{bmatrix} \approx \begin{bmatrix} \beta_2 & \beta_1 - 1 & 0 \\ \alpha & 0 & 0 \\ \beta_2 \bar{l} - \omega & (\beta_1 - 1)\bar{l} & \delta \end{bmatrix} \begin{bmatrix} c - \bar{c} \\ \pi - \bar{\pi} \\ l - \bar{l} \end{bmatrix} \qquad (79)$$

With two predetermined state variables, π and l and one non-predetermined state variable, c, we need two stable and one unstable characteristic root for local stability. One characteristic root, the one driving the real debt dynamics is $\lambda_3 = \delta > 0$. This is really the end of the story, but for completeness, note that the two other eigenvalues (the ones driving c and π) are the same as under the Ricardian fiscal rule given in equation (75), and satisfy equation (77), with one stable and one unstable root if the normal Taylor rule assumptions hold with $\beta_1 > 1$ and $\beta_2 > 0$.

Of course, the model of the economy analyzed here is a simplified version of the Sims (2011) New-Keynesian model and the FFMP just considered is in the spirit of Sims's non-Ricardian FFMP – not identical to it. It is therefore possible that the true Sims (2011) non-Ricardian FFMP was a lucky choice that avoided explosive behavior of the public debt despite the absence of any stabilizing feedback loop from the debt stock to the augmented primary surplus. Even with the stabilizing Taylor rule, I would not recommend (78) to any policy maker, however: there has to be responsiveness of the (augmented) real primary surplus to the real debt stock to prevent the debt dynamics from becoming explosive. Responsiveness of taxes to the level of economic activity ($\omega > 0$) is not sufficient. That responsiveness can be more flexible than what is embodied in the Ricardian rule in equation (75). For instance, the stabilizing feedback of the real augmented primary surplus from the real debt stock might not kick in until the real debt stock exceeds some critical level. An example would be

$$\begin{aligned} \tau &= \omega c,\ \omega > 0 && \text{if } l \le l^* \\ &= r(t)l(t) + \zeta\left(l(t) - l^*\right),\ \zeta > 0 && \text{if } l > l^* \end{aligned}$$

2.3b. 'Financial Repression in the New-Keynesian model

Sims states in Sims (2016a, pages 4 and 5): "*… the existence of stable, unique equilibria under policies that peg the interest rate and leave fiscal effort unresponsive to the level of real debt does not rely on instant, far-sighted adjustments by rational agents. All that is required is a strong wealth effect on consumption and sufficiently rapid response of inflation to demand.*"

In this Subsection I show that instant, far-sighted adjustments by rational agents are not sufficient to ensure *the existence of stable, unique equilibria under policies that peg the interest rate and leave fiscal effort unresponsive to the level of real debt.* I will therefore replace the Taylor rule given in (74) with a nominal interest rate pegging rule:

$$i(t) = \bar{i} \tag{80}$$

Note that this is just a special case of the Taylor rule in equation (74), with $\beta_1 = \beta_2 = 0$ and $\bar{i} = \delta + \pi^*$. Note that the optimizing, forward-looking New-Keynesian consumer does not have a financial wealth effect on consumption: without money in the model, the (forward-looking) consumption function in our New-Keynesian model in equation (29) becomes $C(t) = \delta\left(l(t) + \int_t^\infty e^{-\int_t^v r(u)du}\left(Y(v) - \tau(v)\right)dv \right)$. The apparent financial wealth effect – the term $\delta l(t)$, vanishes when we substitute the State's IBC, holding with equality, $l(t) = \int_t^\infty e^{-\int_t^v r(u)du}\left(\tau(v) - g(v)\right)dv$, into the consumption function, which becomes $C(t) = \delta\left(\int_t^\infty e^{-\int_t^v r(u)du}\left(Y(v) - g(v)\right)dv \right)$ and exhibits debt neutrality or Ricardian equivalence.[10]

Because this model is a special case of the model summarized in equation (79), the debt dynamics will once again be unstable, because the fiscal effort is unresponsive to the level of the public debt. Again, the dynamics of c and π are decoupled from the dynamics of the real debt stock. The eigenvalues driving c and π are the complex conjugate solutions to:

$$\lambda_1 + \lambda_2 = 0 \tag{81}$$

and

$$\lambda_1 \lambda_2 = \alpha \tag{82}$$

These roots are purely imaginary numbers, $\lambda_1 = \iota\sqrt{\alpha}$ and $\lambda_1 = -\iota\sqrt{\alpha}$, where $\iota^2 = -1$. So, consumption and inflation circle around the steady state without ever getting there. The real debt dynamics are, once again, driven by the unstable characteristic root $\lambda_3 = \delta > 0$. Public debt behavior is, again, explosive. This, of course, invalidates the benign 'sovereign IBC' equilibrium assumption on which the (non-explosive) consumption and inflation dynamics are based.

3. Equilibria at the ELB

We now return to the flexible price level model of Section 2.2 and consider equilibria where the economy is at the ELB. To make the point as dramatically as possible, we assume that the economy is permanently at the ELB.

Under the exogenous nominal interest rate rule this requires:

$$i(t) = \bar{i}^M \quad , \ t \geq 0 \tag{83}$$

There is a unique exogenous money stock rule that supports the economy being permanently at the ELB only if there is satiation (zero marginal utility) in real money balances at a finite stock of real money balances at the ELB *and* the utility of holding real money balances declines for real money holdings larger than the satiation level. In that case:

[10] There are two qualifications to the absence of a financial wealth effect in the forward-looking, optimizing household consumption function. First, financial ownership claims to 'outside' assets (assets for which there is no corresponding liability, like physical capital, land and real estate will show up as wealth in the consumption function, even after consolidating the household and State IBCs. Second, if there is money in the model and if there is irredeemability of central bank money, the term $\lim_{v\to\infty} e^{-\int_t^v r(u)d} \frac{M(v)}{P(v)}$ will appear in the consumption function even after consolidation.

$$\frac{\dot{M}(t)}{M(t)} = \overline{\mu} = \overline{i}^{M} - \delta \ , \ t \geq 0$$
$$M(0) = M_0 > 0$$

However, if there is an infinite demand for real money balances when the pecuniary opportunity cost of holding money is zero - as there is with the logarithmic utility function of equation (24) - then, if the price level is positive, an infinite stock of nominal money balances will always be demanded. Even if there is satiation in real money balances at a finite stock of real money balances, but the utility of money remains constant at the satiation level when the stock of real money balances rises above the minimum level at which satiation occurs, the monetary equilibrium condition does not in general yield a unique price level when the nominal money stock is exogenous and the price level is freely flexible.

We again consider the non-Ricardian FFMP $\tau(t) = \overline{z} + \overline{g} - \frac{\sigma_1(t)}{P(t)}$. The utility function (24) has global non-satiation in real money balances, so the demand for real money balances is infinite at the ELB (equation (99)). The only equilibrium conditions that are different at the ELB from what they are away from the ELB are the monetary equilibrium condition (99) and, of course, equation (98), which implies equation (100):

$$\frac{P(t)}{M(t)} = 0 \tag{84}$$

$$\pi(t) = \overline{i}^{M} - \delta \tag{85}$$

Monetary equilibrium requires an infinite stock of real money balances because of the non-satiation feature of the utility function. This can be generated either by a zero price level and a finite nominal stock of money or by a positive price level and an infinite nominal money stock. In principle, at the ELB the nominal money stock can be exogenous (policy-determined) or demand-determined and endogenous.

The direct analogue with the FTPL under interest rate pegging away from the ELB is where the nominal money stock is endogenously determined. If the misspecified equilibrium bond pricing equation (equation (68)) implies a positive price level, we have the FTPL again. Four of the five anomalies that are present with a flexible price level, interest rate pegging and a non-zero stock of nominal government bonds away from the ELB, are present at the ELB also: the price level can be negative; the FTPL cannot determine the price level if there are only index-linked or foreign-currency-denominated government bonds; the logic of the FTPL is no stronger than the logic of the HTPL; a correct specification of the equilibrium bond pricing equation causes the FTPL to disappear. Of course, the 'phlogiston anomaly' – the ability to price money even if money only exists as an abstract numeraire does not exist at the ELB, because there can be no ELB if money does not exist as an asset …

A sticky general price level still makes the FTPL inconsistent at the ELB for the same non-Ricardian FFMPs that were analysed earlier (equations (71) and (73)).

An exogenous and finite nominal money stock is only consistent with monetary equilibrium and a flexible price level if the price level is zero (equation (99)). That too would be inconsistent with price level implied by the misspecified bond pricing equilibrium equation.

The infinite demand for real money balances (equation (99)) at the ELB is implausible both a-priori and empirically. Japan, the Eurozone, Sweden and Denmark have been at the EBL for a significant amount of time, and there has been no evidence of an infinite demand for central bank money in any of these countries. To make sure that the results don't depend on this feature, I (briefly) consider the alternative household utility function below, which exhibits satiation in real money balances when $\frac{M}{P}=\frac{k_2}{k_1}$.

$$
\begin{aligned}
&u(t)=\int_t^{\infty} e^{-\delta(v-t)}\left[\ln C(v)+\xi\left(\frac{M(v)}{P(v)}\right)\right]dv \\
&C(v), M(v)\geq 0;\ \delta,\ \phi\geq 0 \\
&\xi\left(\frac{M}{P}\right)=\varphi\left(-\frac{k_1}{2}\left(\frac{M}{P}\right)^2+k_2\frac{M}{P}\right) \quad \text{if } \ 0\leq\frac{M}{P}\leq\frac{k_2}{k_1} \\
&=\frac{(k_2)^2}{2k_1} \qquad\qquad \text{if } \ \frac{M}{P}>\frac{k_2}{k_1} \\
&C, M\ \ \geq 0;\ k_1,\ k_2,\ \delta,\varphi>0
\end{aligned}
$$

The only thing that changes as a result of this alternative utility function is the demand for real money balances, which becomes:

$$
\begin{aligned}
\frac{M(t)}{P(t)}&=\frac{k_2}{k_1}-\left(\frac{i-\bar{i}^M}{k_1\varphi C(t)}\right) \quad \text{if } \ i(t)>\bar{i}^M \\
&\geq\frac{k_2}{k_1} \qquad\qquad \text{if } \ i(t)=\bar{i}^M
\end{aligned} \tag{86}
$$

The monetary equilibrium condition at the ELB becomes, instead of equation (99):

$$
\frac{M(t)}{P(t)}\geq\frac{k_2}{k_1} \tag{87}
$$

Note that satiation in real money balances at a finite level of real money balances only refers to the non-pecuniary, *direct* utility derived from money balances. Even at the ELB, money remains a store of value and larger money balances make a household better off because wealth is higher. If there is no satiation in consumption (a property of both utility functions), higher holdings of real money balances will boost household demand for consumption and the household IBC will continue to hold with equality.

All but two of the results about the anomalies and the mathematic inconsistencies of the FTPL/FTLEA when the economy is away from the ELB now remain valid when the economy is permanently at the ELB. One result that does not carry over without qualification is that a non-Ricardian FFMP implies an overdetermined model when an exogenous monetary rule setting a finite nominal money stock is followed. As we saw earlier, when there is no satiation in real money balances, the infinite demand for real money balances at the ELB can only be satisfied at a zero general price level, making the FTPL overdetermined, if the nominal money stock is finite. If there is satiation in real money balances at a finite stock of real money balances (equation (102) holds), the system is not necessarily overdetermined under an exogenous money supply rule even at the ELB, because, as long as the exogenous nominal money stock and the (positive) price level determined by the misspecified bond pricing equilibrium condition satisfy equation (102), the monetary equilibrium condition will not determine the price level: the household is indifferent between holding real money balances k_2/k_1 and holding any amount of real money balances greater than k_2/k_1. If, at the price level determined by the misspecified bond pricing equilibrium condition, the real money stock is smaller than k_2/k_1, we cannot be at the ELB.

Of course, the phlogiston anomaly cannot occur because there can be no ELB if money does not exist as a store of value. The other anomalies and the inconsistency when the price level is sticky apply at the ELB also if there is satiation in real money balances at a finite stock of real money balances.

4. Financial repression in an Old-Keynesian model: NOT the FTPL/

In Sims (2016a) an Old-Keynesian model is analyzed. The key difference with the Sims (2011) New-Keynesian model is that consumption is driven by an ad-hoc differential equation and is viewed as a predetermined variable. That means that the boundary condition for consumption is an initial condition: $c(0) = c_0$. If we still impose equation (68), the IBC of the State, holding with equality, as an equilibrium condition, we have an overdetermined system. Sims has stated in private correspondence that equation (68) is not part of the Sims (2016a) model and that, consequently, the Sims (2016a) model is not an FTPL/FTLEA model. With both the household and the State pursuing non-Ricardian programs, there is no presumption that the IBC of the household and of the State will be satisfied in equilibrium, let alone satisfied with equality.

I first analyze a stripped-down version of the Sims (2016a) model. It differs from the actual Sims (2016a) model in two ways. In the Sims (2016a) model, all exogenous variables and policy instruments follow stable, univariate, first-order dynamic systems. I assume instead that all exogenous variables and policy instruments are constant. These constant values can be viewed as the steady-state values of Sims's univariate dynamic processes for these variables. Second, Sims adds additive stochastic shocks to many of the dynamic processes of the model. I leave these out. Nothing essential is lost by these simplifications, but I can now use analytical methods rather than the numerical simulations used by Sims. The stripped-own model has a number of weaknesses that make it unfit as a guide to policy. I therefore developed an alternative Old-Keynesian model that does not share these weaknesses.

The (stripped-down) Sims (2016a) model can be summarized as follows:

$$\begin{aligned}
&\tau = \bar{\tau} = \omega_0 \\
&i = \bar{i} \\
&\pi = \gamma_p\left(v - \ln\theta\right) + \gamma_p\left(\frac{1-\theta}{\theta}\right)c \\
&\dot{c} = \gamma_c\left(l - \frac{\bar{\tau}}{\bar{i}}\right) \\
&\dot{v} = -\gamma_p\left(v - \ln\theta\right) + \left(\frac{\gamma_w - \gamma_p(1-\theta)}{\theta}\right)c \\
&\dot{l} = (i-\pi)l - \tau \\
&\gamma_p, \gamma_w, \gamma_c > 0;\ 0 < \theta < 1
\end{aligned} \tag{88}$$

v is the logarithm of the real wage. Note that there is no responsiveness of the fiscal effort to the public debt or to the level of economic activity, but that the growth rate of consumption responds positively to household financial wealth.

The steady state of the system is given by

$$\begin{aligned} c &= \overline{c} = 0 \\ \pi &= \overline{\pi} = 0 \\ l &= \overline{l} = \frac{\omega_0}{\overline{i}} \\ v &= \overline{v} = \ln\theta \\ r &= \overline{r} = \overline{i} \end{aligned} \tag{89}$$

The linear approximation at the steady state of this system is:

$$\begin{bmatrix} \dot{c} \\ \dot{v} \\ \dot{l} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & \gamma_c \\ \dfrac{\gamma_w - \gamma_p(1-\theta)}{\theta} & -\gamma_p & 0 \\ -\dfrac{\omega_0}{\overline{i}}\gamma_p\left(\dfrac{1-\theta}{\theta}\right) & -\dfrac{\omega_0}{\overline{i}}\gamma_p & \overline{i} \end{bmatrix} \begin{bmatrix} c-\overline{c} \\ v-\overline{v} \\ l-\overline{l} \end{bmatrix} \tag{90}$$

All three state variables are predetermined, so all three eigenvalues must have negative real parts for the system to be stable. The three eigenvalues must satisfy the following conditions:

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= \overline{i} - \gamma_p < 0 \\ \lambda_1\lambda_2\lambda_3 &= -\gamma_c\gamma_p\gamma_w\frac{\omega_0}{\theta\overline{i}} < 0 \end{aligned} \tag{91}$$

Sims assigns a positive steady-state nominal interest rate: $\overline{i} = 0.03$ and very large responsiveness of inflation to aggregate demand: $\gamma_p = 4.00$. The first condition in equation (86) is therefore satisfied. The second condition will then be satisfied if and only if $\omega_0 > 0$: the (steady state) value of the (exogenous) (augmented) real primary surplus is positive, which is the same condition that is necessary for stability in our Old-Keynesian model.

The characteristic equation is

$$\lambda^3 + (\gamma_p - \overline{i})\lambda^2 - \gamma_p\left(\overline{i} - \frac{\gamma_c\omega_0(1-\theta)}{\overline{i}\,\theta}\right)\lambda + \frac{\gamma_c\gamma_w\gamma_p\omega_0}{\overline{i}\,\theta} = 0$$

With the numerical values assigned by Sims, the characteristic equation is

$\lambda^3 + 3.97\lambda^2 + 0.309\lambda + 0.343 = 0$ which indeed has three roots with negative real parts:

$\lambda_1 = -3.9134;\ \lambda_2 = -0.02828 + 0.2947\iota;\ \lambda_2 = -0.02828 - 0.2947\iota$.[11]

Two things must be emphasized about this result. First, it has nothing at all to do with the FTPL/FTLEA. This is a classic fiscal stimulus plus financial repression (a constant nominal interest rate despite permanently higher inflation). The assumption that the IBC of the State holds with equality is not used. The FFMP resembles what has been recommended by some economists to the Japanese authorities: peg the policy rate near zero (in Japan, peg the yield curve near zero for maturities up to 10 years) and provide a (large/long-lasting) deficit-financed fiscal stimulus.

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[11] The numerical values are: $\overline{i} = 0.03; \omega_0 = 0.03; \theta = 0.70; \gamma_w = 0.30; \gamma_c = 2.00; \gamma_p = 4.00$.

Second, this result is not robust, two ways. First, the model is unstable with minor changes in the parameter values - $\omega_0 < 0$ suffices. A shallow slope of the price Phillips curve ($\gamma_p < \bar{i}$) also produces instability. Second, and more important, there are two problems with this model that make it quite inadequate – even at a purely qualitative level - as a guide to the interaction of (nominal) public debt, fiscal policy, nominal interest rate rules and inflation.

First, the price Phillips curve is always upward-sloping (increasing in real aggregate demand) rather than vertical at least in steady state. This means that the inflationary impulse of a cut in taxes, through consumption, will persist as long as the real public debt burden does not start to decline (see the third and fourth equations in (83). With the nominal interest rate pegged, the real interest rate can, for certain parameter values, be reduced enough by enough to run a budget surplus. This permanently upward-sloping price Phillips curve represents a form of permanent inflation illusion that makes the model not fit for purpose.

Second, the reason steady-state inflation is zero in the Sims (2016a) model is that he has the dynamics of real consumption driven by the *nominal* interest rate. Again, I don't consider such permanent inflation illusion to be a desirable property of consumption behavior, even in an Old-Keynesian model. In steady state, real taxes have to equal the real interest rate bill to keep the real public debt constant. Real taxes have to equal the nominal interest rate bill to keep real consumption constant. This certainly pins down the steady-state inflation rate, despite the long-run upward-sloping price Phillips curve, but it really makes no economic sense. If the consumption equation were instead written as $\dot{c} = \gamma_c \left(l - \frac{\bar{\tau}}{\bar{i} - \bar{\pi}} \right)$, that is, the growth rate of consumption depends on the (steady state) real interest rate, there would be infinitely many steady state solutions for $c, v, l,$ π and r. With the nominal interest rate in the consumption function pinning down steady-state inflation at zero and with a long-run upward-sloping Phillips curve, inflation can wipe out the public debt burden during the transition and return safely to zero in the long run. If only.

To get a more robust Old-Keynesian world of Sims (2016a), I will replace the optimizing household/consumer with the 'wealth effect augmented' Keynesian consumption function in equation (87) :

$$\begin{aligned} &C = -\eta_1 \tau + \eta_2 l - \eta_3 r \\ &\eta_1 > 0, \eta_2 > 0, \eta_3 > 0 \end{aligned} \qquad (92)^{12}$$

Consumption depends negatively on real taxes, negatively on the real interest rate and positively on real financial wealth. I will again addition adopt the (unimportant) simplifying assumptions of Sims (2016a), that $\bar{g} = \sigma_1(t) = 0$, so $s + \frac{\sigma_1}{P} = \tau$. The Phillips curve is again an accelerationist one.

$$\begin{aligned} &\dot{\pi} = \alpha(Y - \bar{Y}) \\ &\alpha > 0 \end{aligned} \qquad (93)$$

[12] Because Sims works with a model without money, financial wealth equals the value of the net stock of government bonds held by the households. Outside the phlogiston economy, real financial wealth would be equal to $l + \frac{M}{P}$.

The tax function is:

$$\tau = \omega_0 + \omega_1 l$$
$$\omega_0 < \overline{Y},\ \omega_1 > 0 \tag{94}$$

The model is completed with:

$$C = Y$$
$$r = i - \pi$$
$$\dot{l} \equiv rl - \tau$$

The steady-state equilibrium is given by:

$$\pi = -\left(\frac{\eta_2 - \eta_1\omega_1}{\eta_3}\right)l + \overline{i} + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3}$$
$$\left(\overline{i} - \pi - \omega_1\right)l = \omega_0 \tag{95}$$

Even in this extremely basic model, there are in general two steady-state equilibria. The only way to avoid two steady-state equilibria is to assume that the real interest rate does not affect aggregate demand. In that case, the only way the real interest rate enters the model is through the public debt dynamics. I consider this highly implausible.

$$l = \overline{l} = \frac{\omega_1 + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3} \pm \sqrt{\left(\omega_1 + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2 - \eta_1\omega_1}{\eta_3}\right)\omega_0}}{2\left(\frac{\eta_2 - \eta_1\omega_1}{\eta_3}\right)}$$

$$\pi = \overline{\pi} = -\left(\frac{\eta_2 - \eta_1\omega_1}{\eta_3}\right)\overline{l} + \overline{i} + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3}$$
$$= \frac{-\left(\omega_1 + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right) \pm \sqrt{\left(\omega_1 + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2 - \eta_1\omega_1}{\eta_3}\right)\omega_0}}{2} + \overline{i} + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3} \tag{96}$$

$$r = \overline{r} = \left(\frac{\eta_2 - \eta_1\omega_1}{\eta_3}\right)\overline{l} - \left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)$$
$$= \frac{\left(\omega_1 + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right) \pm \sqrt{\left(\omega_1 + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2 - \eta_1\omega_1}{\eta_3}\right)\omega_0}}{2} - \left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)$$

Note that the steady-state real interest rate is independent of the pegged value of the nominal interest rate, unlike the Sims (2016a) model where the two are equal. I will assume, in the spirit of Sims (2016a), that a higher real stock of public debt boosts consumption demand, even after the effect of a higher public debt on taxes is allowed for, so $\eta_2 - \eta_1\omega_1 > 0$. I also assume that $\overline{Y} + \eta_1\omega_0 > 0$; a sufficient condition for this is $\omega_o > 0$. This means that the real debt stock will be positive in one steady state and negative in the other steady state. In the steady state with the negative level of real public debt, the inflation rate is positive if the pegged nominal interest rate is not too negative. In the steady state with the positive level of real public debt, the inflation rate can be either positive or negative given the a-priori restrictions we have imposed on the parameters. The real interest rate is negative in the steady state with the negative real debt stock.

Eliminating equilibrium consumption, the real interest rate and real tax revenues, the model can be reduced to two first-order differential equations in π and l:

$$\dot{\pi} = \alpha\eta_3\pi + \alpha\left(\eta_2 - \eta_1\omega_1\right)l - \alpha\left(\overline{Y} + \eta_1\omega_0 + \eta_3\overline{i}\right)$$
$$\dot{l} = \left(\overline{i} - \pi - \omega_1\right)l - \omega_0$$

Linearizing this around the unique steady state, we obtain the following dynamic system:

$$\begin{bmatrix}\dot{\pi}\\ \dot{l}\end{bmatrix} \approx \begin{bmatrix}\alpha\eta_3 & \alpha(\eta_2 - \eta_1\omega_1)\\ \overline{l} & \overline{i} - \omega_1 - \overline{\pi}\end{bmatrix}\begin{bmatrix}\pi - \overline{\pi}\\ l - \overline{l}\end{bmatrix} \quad (97)$$

(98)

The characteristic roots are solved for from

$$\begin{aligned}\lambda_1 + \lambda_2 &= \alpha\eta_3 + \overline{i} - \omega_1 - \overline{\pi}\\ \lambda_1\lambda_2 &= \alpha\eta_3\left(\overline{i} - \omega_1 - \overline{\pi}\right) - \alpha\left(\eta_2 - \eta_1\omega_1\right)\overline{l}\end{aligned} \quad (99)$$

Consider the case where the tax function is independent of the real value of the public debt, so $\omega_1 = 0$. The characteristic roots then simplify to:

$$\begin{aligned}\lambda_1 + \lambda_2 &= \alpha\eta_3 + \overline{i} - \overline{\pi}\\ \lambda_1\lambda_2 &= \alpha\eta_3\left(\overline{i} - \overline{\pi}\right) - \alpha\eta_2\overline{l}\end{aligned} \quad (100)$$

Because both π and l are predetermined state variables, we need both roots to have negative real parts for the system to be stable. From the second equation in (95), this requires $\eta_3\overline{r} > \eta_2\overline{l}$. The simplified steady-state equations for the case where $\omega_1 = 0$ are:

$$l = \overline{l} = \frac{\frac{\overline{Y} + \eta_1\omega_0}{\eta_3} \pm \sqrt{\left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2}{\eta_3}\right)\omega_0}}{2\left(\frac{\eta_2}{\eta_3}\right)}$$

$$\pi = \overline{\pi} = -\left(\frac{\eta_2}{\eta_3}\right)\overline{l} + \overline{i} + \frac{\overline{Y}}{\eta_3} = \frac{-\left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right) \pm \sqrt{\left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2}{\eta_3}\right)\omega_0}}{2} + \overline{i} + \frac{\overline{Y} + \eta_1\omega_0}{\eta_3} \quad (101)$$

$$r = \overline{r} = \left(\frac{\eta_2}{\eta_3}\right)\overline{l} - \left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right) = \frac{\left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right) \pm \sqrt{\left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2}{\eta_3}\right)\omega_0}}{2} - \left(\frac{\overline{Y} + \eta_1\omega_0}{\eta_3}\right)$$

If the exogenous component of the tax function is positive, $\omega_0 > 0$, one of the steady-state equilibria will have a negative stock of real debt and an associated negative real interest rate. This condition, similar to the stability condition in the Sims (2016a) model (given in equation (86)) is, however, only necessary but not sufficient for local stability. Consider the following numerical values of the parameters: $\bar{Y} = 1$; $\omega_1 = 0.03$; $\eta_1 = 2.00$; $\eta_2 = 2.00$; $\eta_3 = 0.50$; $\alpha = 2.00$. The steady-state equilibrium with the positive stock of real public debt (which also has a positive real interest rate) is locally unstable ($\lambda_1\lambda_2 = -2.06$). The steady-state equilibrium with the negative stock of real public debt and the associated negative real interest rate is also locally unstable. To make either of both of the steady-state equilibria locally stable, we require fiscal effort responsiveness to the public debt. It can be checked numerically that a large positive value for ω_1 is not sufficient for local stability. To guarantee stability of the debt accumulation process, the fiscal effort has to be able to change sign if the real value of the interest bill changes sign, either because the real interest rate changes sign or because the real debt stock changes sign. Only a flexible Ricardian fiscal effort rule like the one given in equation (75), $\tau(t) = r(t)l(t) + \zeta\left(l(t) - l^*\right),\ \zeta > 0$, will guarantee stable public debt dynamics.

The purpose of this section was not to develop a robust Old-Keynesian model for its own sake. We leave that as an exercise for the reader. The purpose is twofold. First, to make clear that the permanent financial repression model of Sims (2016a) has nothing to do with the FTPL/FTLEA and, second, to show that the fiscal policy effectiveness proposition of Sims (2016a) and his message about debt sustainability is not robust. I summarize this message as ‘don’t worry about the fiscal effort responding to the public debt burden; a sufficiently strong (positive) response of private demand to the real value of the public debt and of inflation to private demand (and to a fiscal stimulus) will take care of debt sustainability’. I do not consider the recommendation to engage in a (permanent) fiscal stimulus, to keep the policy rate pegged (technically forever) and not to have the augmented primary surplus respond intelligently to the public debt burden, to be sensible.

5 The good fiscal theory and aggregate demand

The message of the fiscal theory of seigniorage (FTS) is that, at the ELB or away from the ELB, and regardless of whether the demand for real money balances at the ELB is infinite or finite, it is always possible *with a Ricardian FFMP* to boost nominal aggregate demand by any amount desired through a monetized fiscal expansion. The combined monetary and fiscal authorities are always able to implement an effective helicopter money drop. The only condition that needs to be satisfied for this result to be valid is that a monetized expansion of the balance sheet of the central bank is profitable. This will be the case either if the interest rate on money is below the yield on the central bank’s assets – when we are away from the ELB - and/or if central bank money is irredeemable.

Away from the ELB both drivers of the profitability of central bank balance sheet expansion are operative. At the ELB, only the irredeemability channel is operative. It is, however, sufficient to ensure the effectiveness of a helicopter money drop in stimulating nominal aggregate demand. This combined monetary and fiscal policy effectiveness at the ELB (or away from it) depends in no way on the existence of nominal government bonds and has nothing to do with the FTPL/FTLEA.

5.1 The FTS away from the ELB

Consider attain the IBC of the State (holding with equality) in equation **Error! Reference source not found.** and the household consumption function in equation **Error! Reference source not found.**:

$$\frac{M(t)}{P(t)}+l(t)=\int_t^{\infty} e^{-\int_t^v r(u)du}\left(\tau(v)-g(v)+\frac{\left(i(v)-i^M(v)\right)M(v)}{P(v)}\right)dv+\frac{1}{P(t)}\lim_{v\to\infty} e^{-\int_t^v i(u)du}M(v)$$

$$C(t)=\frac{\delta}{1+\phi}\left(\frac{M(t)}{P(t)}+l(t)+\int_t^{\infty} e^{-\int_t^v r(u)du}\left(Y(v)-\tau(v)\right)dv\right)$$

The two ways in which central bank balance sheet expansion increases fiscal space (relaxes the IBC of the State) and the way in which the State can either use this increased fiscal space to boost public spending on real goods and services or to cut taxes net of transfers are clear from the IBC of the State.

The first fiscal channel associated with a monetized expansion of the central bank balance sheet is through the profits earned by the central bank by issuing liabilities that pay $\bar{i}^M$ and buying assets yielding $i(t)>\bar{i}^N$. Away from the ELB the interest rate on non-monetary instruments exceeds the interest rate on central bank money. The PDV of these profits is $\int_t^{\infty} e^{-\int_t^v r(u)du}\frac{\left(i(v)-i^M(v)\right)M(v)}{P(v)}dv$. These profits accrue to the Treasury, the beneficial owner of the central bank. The Treasury can use these profits either to cut taxes $\tau\left(v\right),\ v\ge t$ and boost household consumption (see the household consumption function) or to boost public spending on goods and services $g(v),\ v\ge t$. These profits disappear, of course, at the ELB.

Note that the State has to make use of the additional fiscal resources made available by a monetized central bank balance sheet expansion for there to be an effect on nominal aggregate demand. So, if the State cuts taxes or increases public spending on real goods and services and monetizes this fiscal stimulus, the potential increase in fiscal space becomes an actual one. If instead the central bank buys government bond debt and, say, holds it forever, rolling it over as it matures, but the government does not use the PDV of the interest saved (or the increase in the PDV of the terminal value of the money stock), to raise public spending or cut taxes – ever – this QE will only increase potential, not actual fiscal space.[13]

The stream of future profits accruing to the State when central bank money pays a lower interest rate than the safe rate on non-monetary financial instruments represents a transfer of resources from the private sector to the State: the term $\int_t^{\infty} e^{-\int_t^v r(u)du}\frac{\left(i(v)-i^M(v)\right)M(v)}{P(v)}dv$ appears with a negative sign in the IBC of the household sector, given in equation (22). This is not the case for the PDV of the terminal stock

[13] If the State satisfies its intertemporal balance sheet exactly and if the state is not liquidity-constrained, the following financial actions of the central bank are equivalent. (1) The central bank buys X amount of *N*-period sovereign debt, paying for it with central bank money, and rolls it over forever when it matures; (2) The central bank buys X amount worth of perpetuities from the State, paying for it with central bank money, and holds them forever; (3) the central bank buys *X* amount of sovereign debt of any maturity, paying for it with central bank money, and cancels it (forgives the debt); and (4) the central bank uses its money to buy X worth of assets from parties other than the State and holds them forever.

of money balances, $\frac{1}{P(t)}\lim_{v\to\infty} e^{-\int_t^v i(u)du} M(v)$, which, because of the irredeemability of central bank money, enters the household IBC as an asset but does not enter the IBC of the State as a liability.

Because of the irredeemability of central bank money, nominal aggregate demand can also be boosted by any amount by raising the PDV of the terminal stock of central bank money, $\lim_{v\to\infty} e^{-\int_t^v i(u)du} M(v)$, by the appropriate amount and either cutting the PDV of current and future taxes by that amount or boosting the PDV of exhaustive public spending by that amount. The effect on real demand will depend on whether nominal prices are flexible or sticky and, if sticky, on how far actual output is below or above potential output.

The IBC of the State shows that an increase in the PDV of the terminal stock of central bank money can be utilized to boost demand either through a cut in the PDV of current and future taxes or through an increase in the PDV of public spending on goods and services. Cuts in 'lump-sum' taxes would be closest to Friedman's original parable of helicopter money drops (Friedman (1956)), but a monetized stimulus to exhaustive public spending (current or capital) is equally feasible.

5.2. The FTS at the ELB

When the economy is permanently at the ELB, the household consumption function, when we substitute the IBC of the State into it, becomes equation (103) if we adopt the utility function with satiation in real money balances at k_2 / k_1.

$$C(t) = \delta\left[\int_t^\infty e^{-\int_t^v r(u)du}\left(Y(v) - g(v)\right)dv + \frac{1}{P(t)}\lim_{v\to\infty} e^{-\bar{i}^M (v-t)} M(v)\right] \qquad (102)$$

As was the case away from the ELB, because of the irredeemability of central bank money, nominal aggregate demand can be boosted by any amount by raising the PDV of the terminal stock of central bank money, $\lim_{v\to\infty} e^{-\bar{i}^M (v-t)} M(v)$, by that amount. If the nominal interest rate on central bank money is zero, for example, $\lim_{v\to\infty} e^{-\bar{i}^M (v-t)} M(v) = \lim_{v\to\infty} M(v)$. A permanent increase at time *t* in the stock of base money (relative to the benchmark) by $\Delta\bar{M}$ will raise the terminal value of the nominal stock of money balances by that amount. Holding constant the current and future path of GDP and of government spending on goods and services, this will boost nominal consumption demand by $\delta\Delta\bar{M}$. These results concerning effective helicopter money drops at the ELB go through when the own interest rate on central bank money is non-zero. Any increase in the money stock (relative to some benchmark) that is followed by growth in the money stock at a proportional rate equal to or greater than the own interest rate on money, will boost aggregate demand. This is clear from the term $\lim_{v\to\infty} e^{-\bar{i}^M (v-t)} M(v) = M(t)\lim_{v\to\infty} e^{-\int_t^v \left(\bar{i}^M - \mu(u)\right)du}$. Note also that helicopter bond drops will not boost nominal aggregate demand, since bonds are assumed to be redeemable, which accounts for the absence of $\frac{1}{P(t)}\lim_{v\to\infty} e^{-\int_t^v i(u)du} l(v)$ from the IBC of the consolidated household sector and State.

6. Conclusion

The fiscal theory of the price revel rests on a fundamental fallacy: the confusion of the IBC of the State with a misspecified equilibrium nominal bond pricing equation. This fundamental fallacy generates a number of internal inconsistencies and anomalies that should have led to the rejection of the FTPL as a logically coherent theory. This has not happened. This paper aims to rectify this error.

The issue is not an empirical one. Neither does it concern the realism of the assumptions that are made to obtain the FTPL. It is about the flawed internal logic of the FTPL.

The FTPL remains internally inconsistent and riven with unacceptable anomalies also when the economy is at the ELB. The attempt by Sims (2011) to extend the FTPL to models with nominal price rigidity is a failure.

Monetary policy has an inevitable fiscal dimension – that has nothing to do with the failure of the FTPL. Central bank money is irredeemable and, except at the ELB, is willingly held even though it is pecuniary-rate-of-return dominated. Central banking therefore should be profitable, not only away from the ELB but even at the ELB. The fiscal theory of seigniorage recognizes that the national Treasury is the beneficial owner of the central bank and that, consequently, a monetized balance sheet expansion by the central bank increases fiscal space. This fiscal space can be filled with tax cuts or higher public spending. Helicopter money is the parable of the fiscal dimension of monetary policy. The active use of concerted monetary and fiscal stimulus can always boost nominal aggregate demand.

The FTPL/FTLEA issue is of more than academic interests. Economists should only ever prescribe or recommend Ricardian fiscal-financial-monetary programs to policy makers - rules for public spending, taxation, interest rates and monetary growth that are designed to satisfy the intertemporal budget constraint of the State identically. Non-Ricardian fiscal-financial-monetary programs don't necessarily lead to sovereign insolvency and default. Instead they could waste fiscal space or even satisfy the IBC of the State with equality in equilibrium. But satisfying the ICB of the State with equality in equilibrium will not be by design, but as the result of good luck. Policy makers convinced of the validity of the FTPL/FTLEA could design and implement fiscal-financial-monetary programs that waste fiscal space or lead to the explosive growth of public debt, followed by some combination of a belated painful fiscal tightening, runaway inflation or even hyperinflation and sovereign default. That is the ethical, practical policy perspective.

The logical, intellectual perspective is that the assertion of the proponents of the FTPL and of the FTLEA that non-Ricardian fiscal-financial-monetary programs are fine because either the price level (in the case of the FTPL) or the level of economic activity (in the case of the FTLEA) will adjust to ensure that the intertemporal budget constraint of the State holds *in equilibrium* even though it does not hold identically/by design, is simply incorrect. The FTPL is false theory. It could also be dangerous. It is time to rebury it.

The fiscal theory of the price level is dead, but the fiscal theory of seigniorage – the rigorous foundation of helicopter money drops - is very much alive.

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