Econometrica, Vol. 52, No. 3 (May, 1984)

SADDLEPOINT PROBLEMS IN CONTINUOUS TIME RATIONAL EXPECTATIONS MODELS: A GENERAL METHOD AND SOME MACROECONOMIC EXAMPLES

BY WILLEM H. BUITER[1]

The paper presents general solution methods for rational expectations models that can be represented by systems of deterministic first order linear differential equations with constant coefficients. One method is the continuous time adaptation of the method of Blanchard and Kahn. To obtain a unique solution there must be as many linearly independent boundary conditions as there are linearly independent state variables. Three slightly different versions of a well-known small open economy macroeconomic model are used to illustrate three fairly general ways of specifying the required boundary conditions. The first represents the standard case in which the number of stable characteristic roots equals the number of predetermined variables. The second represents the case when the number of stable roots exceeds the number of predetermined variables but a sufficient number of linear restrictions on the state variables at an initial date is given to guarantee a unique solution. The third represents the case when the "missing" initial conditions have been replaced by boundary conditions that involve linear restrictions on the values of the state variables across an initial and a future date.

The method of this paper permits the numerical solution of models with large numbers of state variables. Any combination of anticipated or unanticipated, current or future and permanent or transitory shocks can be analyzed.

1. INTRODUCTION

THIS PAPER STUDIES the solution of a class of rational expectations models that can be represented by systems of deterministic first order linear differential equations with constant coefficients. This class includes virtually all deterministic continuous time rational expectations models in the macroeconomic and open economy macroeconomic literature such as Sargent and Wallace [22], Dornbusch [13], Wilson [25], Krugman [16], Dornbusch and Fischer [14], Buiter and Miller [7, 8, 9], Begg [2], and Obstfeld [20]. The two solution methods proposed in the paper handle systems with state vectors of any dimension, n. As long as the forcing variables or exogenous variables do not "explode too fast," any combination of anticipated or unanticipated, current or future, and permanent or transitory shocks can be analyzed. Wilson's [25] analysis of anticipated future shocks in systems where $n = 2$ and Dixit's [12] method for handling unanticipated current permanent shocks are special cases of the general methods developed in this paper.

When the number of predetermined or backward-looking variables (n_1) equals the number of stable roots of the characteristic equation of the homogenous system and the number of non-predetermined, forward-looking or "jump" variables ($n - n_1$) equals the number of unstable roots, there is a natural way of

[1] I would like to thank Marcus Miller, Stephen Turnovsky, and other members of the University of Warwick 1981 Summer Workshop for helpful early comments and suggestions. The comments of two anonymous referees and of the editors on a previous draft helped improve content and exposition. Financial support from the Leverhulme Trust is gratefully acknowledged.

specifying the n linearly independent boundary conditions that are required for a unique solution. This case is considered in the first part of Section 2. n_1 boundary conditions take the familiar form of initial conditions for the predetermined variables. The remaining $n - n_1$ boundary conditions are obtained from the terminal or tranversality condition that the system should be "convergent." More precisely, if a particular solution of the system of equations exists and remains bounded for all time, then the general solution of the system should remain bounded for all time. This transversality condition constrains the initial values of the $n - n_1$ non-predetermined variables to lie on the stable manifold; the influence of the $n - n_1$ unstable characteristic roots is neutralized.[2,3]

If the system has "too many" unstable roots, i.e. if there are fewer stable roots than predetermined variables, no convergent solution exists for arbitrary initial values of the predetermined variables and the methods of this paper cannot be utilized. The case when there are more stable roots than predetermined variables is considered in the second part of Section 2 and in Section 3. The transversality condition that the solution be convergent now no longer suffices to ensure a unique solution. Two examples are given in which economically plausible additional linear boundary conditions can be provided to guarantee uniqueness. One involves linear restrictions on the state variables at the initial date. The other involves linear restrictions on the state variables across an initial and a future date. Formally, all these models can be viewed as linear two-point boundary value problems with linear boundary conditions. The mathematical conditions for uniqueness are straightforward. The problem lies in the economic motivation of the boundary conditions. In *ad hoc* macromodels this motivation can never be fully satisfactory.

In Section 4 three variants of the familiar Dornbusch [**13**] model of an open economy with a freely floating exchange rate are presented. Each variant illustrates one of the three kinds of boundary value problems expounded in Sections 2 and 3.

2. A CONTINUOUS TIME VERSION OF THE METHOD OF BLANCHARD AND KAHN

Case 1: *As Many Predetermined Variables as Stable Roots*

We define the following notation. x is an n_1 vector of predetermined variables, i.e. $x(t_0)$ is given. y is an $n - n_1$ vector of non-predetermined variables; unlike

[2] See Brock [**6**] for a model in which these transversality conditions are derived from explicit optimizing behavior by an infinite-lived consumer. The non-predetermined variables there have the interpretation of co-state variables in a dynamic optimization problem.

[3] The non-predetermined variables frequently are asset prices determined in efficient asset markets. Arbitrage conditions or, more generally, optimal intertemporal speculation, are assumed to rule out anticipated *future* jumps in these asset prices. Thus, except at those instants at which new information arrives, the non-predetermined variables are continuous functions of time. See Calvo [**11**].

$x(t)$, $y(t)$ can respond, at time t, to changes in the information set conditioning expectations formed at time t.[4] The $n - n_1$ boundary conditions for $y(t)$ take the form of terminal or transversality conditions to be determined below. z is a k vector of exogenous or forcing variables. A and B are constant matrices. Let E be the expectation operator and $I(t)$ the information set conditioning expectations formed at time t. For any vector w,

$$\dot{w}(t) \equiv \lim_{s \downarrow t} \frac{w(s) - w(t)}{s - t}; \quad E_t w(s) \equiv E(w(s) \mid I(t)) \quad \text{and}$$

$$E_t \dot{w}(s) = E\left(\lim_{u \downarrow s} \frac{w(u) - w(s)}{u - s} \middle| I(t) \right).$$

Consider the linear model given in equation (1):

$$\text{(1)} \qquad \begin{bmatrix} \dot{x}(t) \\ E_t \dot{y}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + Bz(t).$$

The information set $I(t)$ contains all current and past values of x, y, and z, as well as the true structure of the model given in (1). This implies

$$\text{(2a)} \qquad E_t y(s) = y(s), \qquad s \leq t.$$

For $s < t$ this means "perfect hindsight"; for $s = t$ it is the assumption of "weak consistency" made for example in Turnovsky and Burmeister [24].

We also assume

$$\text{(2b)} \qquad I(t) \supseteq I(s), \qquad t > s.$$

We rule out explosive growth of the expectation, held at time t, of future values of z by assuming that $E_t z(s)$ is a bounded function of s on $[t, +\infty)$ and continuous almost everywhere.

From the weak consistency assumption it follows that

$$\text{(3)} \qquad \dot{w}(t) = E_t \dot{w}(t) + \lim_{s \downarrow t} \frac{E_s w(s) - E_t w(s)}{s - t}.$$

The second term on the right-hand side of (3) is the instantaneous rate at which expectations of $w(t)$ are revised. Equation (1) implies that

$$\text{(4)} \qquad \dot{x}(t) = E_t \dot{x}(t) \quad \text{or} \quad \lim_{s \downarrow t} \frac{E_s x(s) - E_t x(s)}{s - t} = 0.$$

The actual and anticipated rates of change of the predetermined variables coincide. This need not be true for the non-predetermined variables, y. When

[4] See Buiter [10].

new information becomes available, $\lim_{s\downarrow t}(E_s y(s) - E_t y(s))/(s-t)$ can become non-zero and y may even move discontinuously at such instants.[5,6]

Taking expectations of (1) conditional on $I(t)$ yields

$$\begin{bmatrix} E_t\dot{x}(t) \\ E_t\dot{y}(t) \end{bmatrix} = A\begin{bmatrix} E_t x(t) \\ E_t y(t) \end{bmatrix} + BE_t z(t). \tag{5}$$

We assume that A can be diagonalized by a similarity tranformation as in (6):

$$A = V^{-1}\Lambda V \quad \text{or} \quad VAV^{-1} = \Lambda. \tag{6}$$

V is an $n \times n$ matrix whose rows are linearly independent left-eigenvectors of A. Λ is a diagonal matrix whose diagonal elements are the characteristic roots of A. We make the following assumption.

ASSUMPTION (A1): A has n_1 characteristic roots with non-positive real parts ("stable" roots) and $n - n_1$ characteristic roots with positive real parts ("unstable" roots).

A, B, V, V^{-1} and Λ are partitioned conformably with x and y:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix};$$

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}; \qquad V^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}; \qquad \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}.$$

Λ_1 is an $n_1 \times n_1$ diagonal matrix containing the stable roots of A and Λ_2 an $(n - n_1) \times (n - n_1)$ diagonal matrix containing the unstable roots of A.

[5] This characterization is very similar to that obtained by Gray and Turnovsky **[15]**, who describe this type of relationship in terms of the differentiability properties of the variable being forecast.

[6] The analysis in the paper is for deterministic linear systems. One should therefore interpret the expectations as held with complete subjective certainty. The results are, however, fully transferable to the case of stochastic linear differential equation systems such as

$$\begin{pmatrix} dx(t) \\ E_t dy(t) \end{pmatrix} = A\begin{pmatrix} x(t)\,dt \\ y(t)\,dt \end{pmatrix} + Bz(t)\,dt + dv.$$

$z(t)$ is a measurable and bounded strictly deterministic function of time and the vector time series $v(t)$ is a stationary stochastic process with independent increments and zero mean. Examples are Gaussian white noise (Wiener processes or Brownian motion) when successive increments are normally distributed, Poisson white noise (Jump processes) when increments have a Poisson distribution and mixed Poisson and Brownian processes. By switching to differential notation, adding a white noise disturbance term and interpreting integrals as stochastic (e.g. Ito) integrals, the analysis of the linear deterministic case can be made to apply to the linear stochastic case, provided the weak consistency condition (2a) is satisfied.

Define

$$(7) \qquad \begin{pmatrix} p \\ q \end{pmatrix} = V\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = V^{-1}\begin{pmatrix} p \\ q \end{pmatrix}.$$

p is an n_1 vector and q an $n - n_1$ vector.

From (5), (6), and (7) we obtain

$$(8) \qquad E_t\dot{q}(t) = \Lambda_2 E_t q(t) + (V_{21}B_1 + V_{22}B_2)E_t z(t).$$

Let

$$(9) \qquad D = V_{21}B_1 + V_{22}B_2 .$$

If u, v, and w are random vectors with joint density $f(u, v, w)$, then $E(E(u \mid v, w) \mid w) = E(u \mid w)$. Therefore, since $I(s) \supseteq I(t)$, $s \geq t$, equation (8) implies

$$(10) \qquad E_t\dot{q}(s) = \Lambda_2 E_t q(s) + DE_t z(s), \qquad s \geq t.$$

The forward-looking solution for $E_t q(s)$ is

$$(11) \qquad E_t q(s) = e^{\Lambda_2 s}K_2 - \int_s^\infty e^{\Lambda_2(s-\tau)}DE_t z(\tau)\, d\tau, \qquad s \geq t.^7$$

K_2 is an $n - n_1$ vector of arbitrary constants. Given our assumptions about $E_t z(\tau)$ the integral on the right-hand side of (11) exists. Equation (11) will converge only if $K_2 = 0$. This is the "transversality condition" that has become standard in linear rational expectations models. This choice of the convergent solution can sometimes be rationalized in terms of the transversality conditions characterizing an optimal intertemporal plan in models with infinite-lived households (Brock [**6**]). In most macroeconomic models the justification for requiring the non-predetermined variables (usually asset prices) to behave like the co-state variables of Hamiltonian dynamics, is *ad hoc*. Given $K_2 = 0$, (11) becomes

$$(12) \qquad E_t q(s) = -\int_s^\infty e^{\Lambda_2(s-\tau)}DE_t z(\tau)\, dt.$$

[7] The exponential matrix e^C where C is an $n \times n$ matrix is defined by $e^C \equiv \sum_{k=0}^{\infty}(C^k/k!)$. When C is a diagonal matix,

$$C = \begin{bmatrix} c_i & & & & 0 \\ & \ddots & & & \\ & & c_i & & \\ & & & \ddots & \\ 0 & & & & c_n \end{bmatrix}, \quad \text{then} \quad e^C \equiv \begin{bmatrix} e^{c_1} & & & & 0 \\ & \ddots & & & \\ & & e^{c_i} & & \\ & & & \ddots & \\ 0 & & & & e^{c_n} \end{bmatrix}.$$

Evaluating this at $s = t$ and using the weak consistency assumption $E_t q(t) = q(t)$, we get

(13) $$q(t) = -\int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau)\, d\tau.$$[8]

From (7) we know that $q = V_{21}x + V_{22}y$. Therefore, provided V_{22} has an inverse,

(14) $$y(t) = -V_{22}^{-1}V_{21}x(t) - V_{22}^{-1}\int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau)\, d\tau.$$

It is easily checked that, provided W_{11} has an inverse, (14) can also be written as

(14′) $$y(t) = W_{21}W_{11}^{-1}x(t) - V_{22}^{-1}\int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau)\, d\tau.$$

The similarity between equation (14) or (14′) and Blanchard and Kahn's [5] equation (3) is apparent. The current value of the non-predetermined variables depends on the current values of the predetermined variables and on current anticipations of all future values of the forcing variables.

To find the solution for $x(t)$ we substitute (14) into (1). This yields

$$\dot{x}(t) = (A_{11} - A_{12}V_{22}^{-1}V_{21})x(t) + B_1 z(t)$$
$$- A_{12}V_{22}^{-1}\int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau)\, d\tau$$

or

(15) $$\dot{x}(t) = W_{11}\Lambda_1 W_{11}^{-1}x(t) + B_1 z(t) - A_{12}V_{22}^{-1}\int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau)\, d\tau.$$

[8] Another way of interpreting the boundary condition $K_2 = 0$ is as follows. (I am indebted to an editor for pointing this out to me.)

$$\frac{d}{ds} E_t(e^{-\Lambda_2 s}q(s)) = E_t(e^{-\Lambda_2 s}(\dot{q}(s) - \Lambda_2 q(s))).$$

For $t \leq s$,

$$\frac{d}{ds} E_t(e^{-\Lambda_2 s}q(s)) = E_t e^{-\Lambda_2 s}(E_s\dot{q}(s) - \Lambda_2 E_s q(s)) = E_t e^{-\Lambda_2 s} D E_s z(s).$$

Integrating forward yields

$$E_t e^{-\Lambda_2 t}q(t) = -\int_t^{\infty} e^{-\Lambda_2 \tau} D E_t z(\tau)\, ds \quad \text{provided}$$

$$\lim_{s\to\infty} E_t(e^{-\Lambda_2 s}q(s)) = 0.$$

This condition is equivalent to $K_2 = 0$. Weak consistency then yields

$$q(t) = -\int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau)\, ds.$$

For the predetermined variables we choose the backward-looking solution. Given an initial condition for the n_1 vector x at $t = t_0$:

(16) $$x(t_0) = \bar{x}(t_0),$$

the solution for $x(t)$ is found to be

(17) $$x(t) = W_{11}e^{\Lambda_1(t-t_0)}W_{11}^{-1}\bar{x}(t_0) + \int_{t_0}^{t} W_{11}e^{\Lambda_1(t-s)}W_{11}^{-1}B_1 z(s)\,ds - \int_{t_0}^{t} W_{11}e^{\Lambda_1(t-s)}W_{11}^{-1}A_{12}V_{22}^{-1}\int_s^{\infty} e^{\Lambda_2(s-\tau)}DE_s z(\tau)\,d\tau\,ds$$

or

(17′) $$x(t) = W_{11}e^{\Lambda_1(t-t_0)}W_{11}^{-1}\bar{x}(t_0) + \int_{t_0}^{t} W_{11}e^{\Lambda_1(t-s)}W_{11}^{-1}B_1 z(s)\,ds - \int_{t_0}^{t} W_{11}e^{\Lambda_1(t-s)}\{\Lambda_1 V_{12}V_{22}^{-1} + W_{11}^{-1}W_{12}\Lambda_2\} \times \int_s^{\infty} e^{\Lambda_2(s-\tau)}DE_s z(\tau)\,d\tau\,ds.$$ [9]

The similarity between (17) or (17′) and Blanchard and Kahn's final form solution for $x(t)$ in their equation (4) is again apparent. The value of the predetermined variables in period t depends on the initial condition $\bar{x}(t_0)$. The influence of the initial conditions vanishes as $t \to \infty$ if Λ_1 contains only roots with negative real parts and remains bounded even if Λ_1 contains roots with zero real parts. $x(t)$ also depends on the *actual* values of the forcing variables between time t_0 and t. Finally it depends on the expectations, formed at each instant s between time t_0 and t, of all values of the forcing variables beyond s.

Dixit's formula.

Consider the special case when the anticipated future values of z are all constant, i.e. $E_t z(\tau) = \bar{z}$, $\tau \geq t$. Equation (14) then simplifies to

$$y(t) = -V_{22}^{-1}V_{21}x(t) - V_{22}^{-1}\Lambda_2^{1}D\bar{z}.$$

Let $\bar{x}$ and $\bar{y}$ be the steady state values of x and y respectively, corresponding to $\bar{z}$. A little manipulation then shows that

(18) $$y(t) - \bar{y} = -V_{22}^{-1}V_{21}(x(t) - \bar{x})$$

or, using (17′),

(18′) $$y(t) - \bar{y} = W_{21}W_{11}^{-1}(x(t) - \bar{x}).$$

[9] Using $e^{W_{11}\Lambda_1 W_{11}^{-1}} = W_{11}e^{\Lambda_1}W_{11}^{-1}$.

These are the formulae obtained by Dixit **[12]** for calculating the effect on the non-predetermined variables of previously unanticipated, immediate, permanent changes in the exogenous variables.

Case 2: Fewer Predetermined Variables than Stable Roots; Linear Restrictions on the Initial Conditions

There are several models of economic interest in which the number of predetermined variables is less than the number of stable roots but in which a unique solution is nevertheless guaranteed through suitable linear restrictions on the initial conditions (e.g., Buiter and Miller **[8, 9]**, Begg and Hague **[3]**). The solution method for Case 1 can be extended straightforwardly to handle this class of models.

Consider equation (1) again. A has n_1 stable roots and $n - n_1$ unstable roots. We partition the n_1 vector x into an n_1' vector x' of predetermined variables and an $n_1 - n_1'$ vector x'' of state variables for which the boundary conditions take the form of linear restrictions on the initial conditions:

$$x = \begin{bmatrix} x' \\ x'' \end{bmatrix}, \tag{19}$$

$$x'(t_0) = \bar{x}'(t_0), \tag{20a}$$

$$F_1 x''(t_0) + F_2 x'(t_0) + F_3 y(t_0) = f. \tag{20b}$$

F_i, $i = 1, 2, 3$, are constant matrices and f is an $n_1 - n_1'$ constant vector. Provided (20b) represents $n - n_1'$ independent boundary conditions, i.e. provided F_1 is invertible, a unique convergent solution exists to the system (1) with boundary conditions (20a, b). It is given by (14), (17), and

$$x(t_0) = \begin{bmatrix} \bar{x}'(t_0) \\ -F_1^{-1}(F_2\bar{x}'(t_0) + F_3 y(t_0) - f) \end{bmatrix}. \tag{21}$$

3. A LINEAR TWO POINT BOUNDARY VALUE PROBLEM: THE METHOD OF ADJOINTS

In this section we consider a solution method for a second class of linear rational expectations models characterized by fewer predetermined variables than stable roots. For this class of models a unique solution can be generated through suitable linear restrictions on the values of the state variables at an initial date, t_0, and a finite terminal date, $t_1 \geq t_0$ (see Minford **[19]** and Taylor **[23]** for a stochastic analogue of this problem).

We consider the model of equation (1) over a time interval $t_0 \leq t \leq t_1$ during which the information set does not change, i.e. $I(t) = I$, $t_0 \leq t \leq t_1$. Over this interval, therefore, $E_t\dot{y}(t) = \dot{y}(t)$ and equation (1) can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + Bz(t), \qquad t_0 \leq t \leq t_1. \tag{22}$$

We now consider the two-point boundary value problem of equations (22) and (23):

$$M\begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} + N\begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix} = r. \tag{23}$$

Equation (23) gives n linear restrictions on the value of the state vector at two distinct dates. M and N are $n \times n$ constant matrices and r is a n-vector of constants. Let

$$w = \begin{bmatrix} x \\ y \end{bmatrix}, \quad M \equiv \{\mu_{ji}\}, \quad N = \{\nu_{ji}\}, \quad \text{and}$$

$$r = (\rho_1, \ldots, \rho_j, \ldots, \rho_n)^{T10} \qquad (i, j = 1, 2, \ldots, n).$$

We can therefore rewrite (23) as (24):

$$\sum_{i=1}^{n} \mu_{ji} w_i(t_0) + \sum_{i=1}^{n} \nu_{ji} w_i(t_1) = \rho_j \qquad (j = 1, 2, \ldots, n). \tag{24}$$

w_i denotes the ith elements of w, $i = 1, 2, \ldots, n$. Consider the adjoint system to (22):

$$\dot{s}(t) = -A^T s(t). \tag{25}$$

We integrate the adjoint equations backward from $t = t_1$, once for each $w_i(t_1)$ in (24), using as the terminal boundary conditions

$$s_i^{(j)}(t_1) = \nu_{ji} \qquad (i, j = 1, 2, \ldots, n). \tag{26}$$

$s_i^{(j)}(t_1)$ is the ith component at $t = t_1$ for the jth backward integration of the adjoint equation. Thus, if ν_j^T denotes the transpose of the jth row of N in equation (24), we have the solution

$$s^{(j)}(t) = e^{-(t-t_1)A^T} \nu_j^T \qquad (j = 1, 2, \ldots, n). \tag{27}$$

Setting $t = t_0$ in (27) we obtain $s^{(j)}(t_0)$.

[10] For any matrix Ω, Ω^T denotes the (complex conjugate) transpose of Ω.

The fundamental identity for the method of adjoints is (see Roberts and Shipman [**21**, pp. 17–22]):

$$(28)\qquad \sum_{i=1}^{n} s_i^{(j)}(t_1)w_i(t_1) - \sum_{i=1}^{n} s_i^{(j)}w_i(t_0) = \int_{t_0}^{t_1}\sum_{i=1}^{n} s_i^{(j)}(t)b_i z(t)\,dt$$

$$(j = 1,2,\ldots,n).$$

b_i is the ith row of the matrix B. Substituting for $s_i^{(j)}(t_1)$ from (26) into (22) and using (24) yields

$$\rho_j - \sum_{i=1}^{n} \mu_{ji}w_i(t_0) - \sum_{i=1}^{n} s_i^{(j)}(t_0)w_i(t_0) = \int_{t_0}^{t_1}\sum_{i=1}^{n} s_i^{(j)}(t)b_i z(t)\,dt$$

$$(j = 1,2,\ldots,n)$$

or

$$(29)\qquad \sum_{i=1}^{n}\left[\mu_{ji} + s_i^{(j)}(t_0)\right]w_i(t_0) = \rho_j - \int_{t_0}^{t_1}\sum_{i=1}^{n} s_i^{(j)}(t)b_i z(t)\,dt$$

$$(j = 1,2,\ldots,n).$$

Equation (29) constitutes a set of n equations in the n unknowns $w_i(t_0)$, $i = 1,2,\ldots,n$. If they are linearly independent they will yield a unique solution for

$$w(t_0) = \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix}.$$

Given the value of the entire state vector at $t = t_0$, equation (22) can be solved as a standard initial value problem. Its solution is

$$(30)\qquad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{A(t-t_0)}\begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} + \int_{t_0}^{t} e^{A(t-s)}Bz(s)\,ds, \qquad t_0 \le t \le t_1.$$

4. AN ILLUSTRATION

Three variants of a well-known open economy macroeconomic model of Dornbusch [**14**] will serve to illustrate the solution methods of the two previous sections. They demonstrate that these methods can be applied to models of economic interest. A detailed discussion of the models and an informal, mainly diagrammatic discussion of their solution can be found in Buiter and Miller [**7**, **8**, **9**]. The first discussion of anticipated future shocks in this class of model can be found in Wilson [**25**]. Equations (31)–(35) and (37)–(38) are common to all three variants:

(31) $$m(t) - p(t) = kq(t) - \lambda r(t), \qquad k, \lambda > 0,$$

(32) $$q(t) = -\gamma(r(t) - E_t \dot{p}(t)) + \delta(e(t) + p^*(t) - p(t)), \qquad \gamma, \delta > 0,$$

(33) $$E_t \dot{e}(t) = r(t) - r^*(t),$$

(34) $$p(t) = \alpha w(t) + (1 - \alpha)(e(t) + p^*(t)), \qquad 0 \leq \alpha \leq 1,$$

(35) $$\dot{w}(t) = \phi q(t) + \pi(t), \qquad \phi > 0,$$

(36a) $$\pi(t) = \dot{m}(t),$$

or

(36b) $$\dot{\pi}(t) = \eta(\dot{p}(t) - \pi(t)), \qquad \eta > 0,$$

or

(36c) $$\pi(t) = \dot{p}(t),$$

(37) $$l \equiv m - w,$$

(38) $$c \equiv e + p^* - w.$$

m is the nominal money stock, p the domestic price level, p^* the exogenous foreign price level, q real output, r the domestic nominal interest rate, e the exchange rate (domestic currency price of foreign currency), w the money wage, π the underlying or "core" rate of inflation, r^* the exogenous world interest rate. All variables except r, r^*, and π are in logs. The LM curve, equation (31), specifies the demand for real money balances as a decreasing funciton of the nominal interest rate and an increasing function of real income. The IS curve, equation (32), gives the demand for output as a decreasing function of the real interest rate and an increasing function of the relative price of imports. The international interest differential equals the expected depreciation of the exchange rate (equation (33)). The domestic price level is a mark-up on domestic unit wage costs and unit import costs (equation (34)). The rate of change of money wages is governed by an augmented wage Phillips curve (equation (35)). The constant level of capacity or full-employment output is set equal to zero. Equations (37) and (38) define two convenient state variables; $l \equiv m - w$ is a measure of real liquidity and $c \equiv e + p^* - w$ is a measure of competitiveness.

In all three variants the exchange rate, e, is treated as a non-predetermined variable. It can make discrete, discontinuous jumps in response to "news." Efficient arbitrage and speculation by risk-neutral foreign exchange operators (equation (33)) rules out anticipated future discontinuous jumps in e. The three variants differ in their treatment of the money wage, w, and in their specification of the behavior of core inflation, π.

Like the original Dornbusch model, the first two examples have a "Keynesian" predetermined money wage. They differ in that the first variant specifies that "core inflation"—the augmentation term in the wage Phillips curve—equals the current rate of growth of the nominal money stock. The second example assumes more inertia in the core inflation process and specifies π as an exponentially declining distributed lag function on current and past inflation. The third example is a "classical" variant of the Dornbusch model. The money wage is no longer viewed as predetermined and core inflation equals the current actual rate of inflation. The first example fits the solution method of Section 2 when the number of predetermined variables equals the number of stable roots. The second example involves boundary conditions in the form of linear restrictions on the state variables at the initial date. The last variant represents the two point boundary value problem of Section 3 where the boundary conditions involve linear restrictions on the state variables at an initial and a future date.

Variant 1: *Money Wage Predetermined*: $\pi = \dot{m}$

In this "Keynesian" variant, the money wage (and therefore l) is treated as predetermined. Core inflation is modelled by equation (36a) as equal to the current rate of growth of the nominal money stock. It is easily checked that there exists a state-space representation of the model involving just two state variables, l and c and three forcing variables, $\dot{m}(t)$, $r^*(t)$, and $\dot{p}^*(t)$. With w predetermined and e non-predetermined, variant 1 can be expressed as follows in terms of the notation of the general solution method developed in Section 2: $l = x$, $c = y$, and $[\dot{m}, r^*, \dot{p}^*]^T = z$. The initial condition for l is $l(t_0) = \bar{l}(t_0) \equiv \overline{m}(t_0) - \overline{w}(t_0)$. A unique convergent solution exists if the state matrix of $[l\ c]$ has one stable and one unstable characteristic root. It is shown in Buiter and Miller [**8**, **9**] that a necessary and sufficient condition for such a saddlepoint equilibrium is $\alpha\gamma(\lambda\phi - k) - \lambda < 0$. The interpretation of this condition is that, at a given level of competitiveness, an exogenous increase in aggregate demand raises output. Provided this condition is satisfied, the solution method of Section 2 can be used to solve for the behavior of the system under any combination of anticipated and unanticipated, current and future, permanent and transitory shocks in the forcing vector z.

Variant 2: *Money Wage Predetermined*: $\dot{\pi} = \eta(\dot{p} - \pi)$

In this second "Keynesian" model, the money wage is again treated as predetermined, but core inflation now follows the familiar adaptive process (36b). There exists a state-space representation of the model involving three state variables, l, π, and c and three forcing variables, $\dot{m}$, r^*, and $\dot{p}^*$. Equation (36b) can be rewritten as

$$\dot{\pi} = \eta((1 - \alpha)\dot{c} + \dot{w} - \pi)$$

or

$$\frac{d}{dt}(\pi - \eta((1 - \alpha)c + w)) = -\eta\pi. \tag{39}$$

Since e is non-predetermined, p and c will move discontinuously whenever e does. Consequently π cannot be treated as predetermined. $\pi - \eta((1-\alpha)c + w)$ can be taken as predetermined, however.[11] For any variable n let $n(t^-) \equiv \lim_{\tau\uparrow t} n(\tau)$. At any instant t, therefore, the following relationship must hold:

$$(40) \qquad \pi(t) = \pi(t^-) + \eta(1-\alpha)(c(t) - c(t^-)).$$

In terms of the notation of the general solution method of Section 2, variant 2 can be expressed as follows: $l = x'$, $\pi = x''$, $c = y$, and $[\dot{m}, r^*, \dot{p}^*]^T = z$. The initial condition for l is the same as in variant 1: $l(t_0) = \bar{l}(t_0) \equiv \overline{m}(t_0) - \overline{w}(t_0)$. The linear restriction on π at $t = t_0$ is obtained by evaluating (40) at $t = t_0$. In terms of equation (20b), $F_1 = 1$, $F_2 = 0$, $F_3 = -\eta(1-\alpha)$, and $f = \pi(t_0^-) - \eta(1-\alpha)c(t_0^-)$. $\pi(t_0^-)$ and $c(t_0^-)$ are data at $t = t_0$.

Given these initial conditions, a unique convergent solution exists if the state matrix of $[l \;\pi\; c]^T$ has two stable and one unstable characteristic roots. For a fairly wide range of plausible values of the parameters, the state matrix of this model will have two stable (complex conjugate) roots and one unstable root. In a recent paper (Buiter and Miller [9]), the central example uses the parameter values $\lambda = 2$, $k = 1$, $\alpha = .75$, $\phi = \eta = \delta = \gamma = .5$. This yielded characteristic roots of .375 and $-.1875 \pm .2977i$. Varying η between .5 and 5 and ϕ between .5 and 2 did not alter the number of stable and unstable roots; neither did setting γ, the interest semi-elasticity of demand, equal to zero.

It is important to note that the two boundary conditions $l(t_0) = \bar{l}(t_0)$ and (40) are part of the complete specification of the model, regardless of what the values of the characteristic roots turn out to be. If for example the state matrix had more than one unstable root, this would imply that the model did not possess a convergent solution. If one believes that real-world economic systems are convergent, i.e. that small perturbations from any given "reference" trajectory do not lead to cumulative, explosive deviations from that trajectory, then instability of a model is evidence of misspecification. We cannot then arbitrarily discard (40) and treat $\pi(t)$ as a "free", non-predetermined variable. In general, the number of stable roots varies with the values of the parameters, e.g. in the central example of Buiter and Miller [9] setting $\phi = 7$ will lead to too many unstable roots and to explosive behavior. The possibility of "too many" (or "too few") unstable roots always exists.

Variant 3: *Money Wage Non-predetermined*: $\pi(t) = \dot{p}(t)$

In this "classical" variant of the model the money wage is no longer treated as predetermined. Core inflation is identified with the actual rate of inflation (36c); complete instantaneous indexation could be a reason for this. The state-space representation of the model involves the two state variables l and c and the three

[11] The model can be expressed with l, c, and Ω (or e, w, and Ω) as state variables where $\Omega \equiv \pi - \eta p$. Equation (36b) can then be written as $\dot{\Omega} = -\eta\Omega - \eta^2 p = -\eta\Omega - \eta^2\alpha w - \eta^2(1-\alpha)(e + p^*)$. This is not a very convenient specification, however, as a stationary equilibrium with $\dot{m} \neq 0$ will not have a constant value of Ω.

forcing variables $\dot{m}$, r^*, and $\dot{p}^*$. For constant values of the forcing variables we can write the equations of motion in terms of deviations of the state variables from their stationary equilibrium values $\bar{l}$ and $\bar{c}$ as follows:

$$\begin{bmatrix} \dot{l}(t) \\ E_t\dot{c}(t) \end{bmatrix} = \begin{bmatrix} \lambda^{-1} & a_{12} \\ 0 & -\phi\alpha\delta(1-\alpha(1+\gamma\phi))^{-1} \end{bmatrix} \begin{bmatrix} l-\bar{l} \\ c-\bar{c} \end{bmatrix}. \tag{41}$$

Note that the model is recursive. The two characteristic roots of the A matrix in (41) are $\rho_1 = -\phi\alpha\delta(1-\alpha(1+\phi\gamma))^{-1}$ and $\rho_2 = \lambda^{-1}$. First I shall argue that both l and c can reasonably be viewed as non-predetermined in this model. For $l \equiv m - w$ this follows immediately from the assumption that w is non-predetermined. Buiter and Miller [**8, 9**] contain discussions of this model for the special case $\alpha = 1$. In this case the two roots, λ^{-1} and δ/γ, are both positive. On the assumption that neither l nor c are predetermined, a unique convergent solution exists. With a flexible money wage and a freely floating exchange rate, the designation of both l and c as non-predetermined appears acceptable and even natural. There would seem to be no reasonable grounds for changing this designation as α becomes positive and the exchange rate is allowed a direct effect on the domestic price level. As long as ρ_1 is positive, a unique convergent solution will exist when c is non-predetermined. This root will, however, be negative if $1-\alpha(1+\gamma\phi) > 0$. To interpret this condition, add a "demand shock" term g on the right-hand side of the IS equation (32). A little manipulation yields:

$$y = \frac{-\gamma(1-\alpha)}{1-\alpha(1+\gamma\phi)}(r^* - \dot{p}^*) + \frac{\alpha(1-\alpha)\delta}{1-\alpha(1+\gamma\phi)}c + \frac{(1-\alpha)}{1-\alpha(1+\gamma\phi)}g.$$

For $0 \leq \alpha < 1$, $1-\alpha(1+\gamma\phi)$ must be positive for an exogenous increase in demand to raise output at a given level of competitiveness. If this condition is satisfied, ρ_1, the characteristic root governing c, is negative. With one positive and one negative root and two non-predetermined state variables the transversality condition that c and l remain bounded for bounded values of the forcing variables no longer suffices to select a unique solution trajectory. At $t = t_0$ there is a continuum of initial values for l and c (a line in $l-c$ space) consistent with a convergent solution.[12] Minford [**19**] has argued that a further boundary condition can be imposed on models such as the present one when $\rho_1 < 0$ which will ensure a unique solution. In the present model, the proposed boundary condition is the restriction that if the forcing variables assume constant values, the state variables assume their corresponding stationary values. Clearly if both ρ_1 and ρ_2 are

[12]Taylor [**23**] encountered a similar "non-uniqueness" problem in discrete time, stochastic linear models with rational expectations of future endogenous variables. He proposed a minimum variance criterion to resolve the non-uniqueness problem in certain cases. Our approach, choosing the solution that achieves the fastest sustainable convergence to the new stationary equilibrium, is in the same spirit (and indeed equally arbitrary). I am indebted to a referee for bringing this to my attention.

positive, the only convergent solution when $\dot{m}(t)$, $r^*(t)$, and $\dot{p}^*(t)$ take on the constant values $\bar{\dot{m}}$, $\bar{r}^*$, and $\bar{\dot{p}}^*$ for $t \geq t_1$ is for l and c to take on the stationary equilibrium values corresponding to $\bar{\dot{m}}$, $\bar{r}^*$, and $\bar{\dot{p}}^*$ at t_1 and beyond. Imposing this stationarity property as a boundary condition when $\rho_1 < 0$ amounts to asserting that this property of the solution should be preserved; a change in the sign of ρ_1 from positive to negative should not render the behavior of this system with its two "forward-looking" non-predetermined variables dependent on "irrelevant" past values of the forcing variables.

The proposed boundary conditions are given in equation (42), assuming that $[\dot{m}(t), \bar{r}^*(t), \dot{p}^*(t)] = [\bar{\dot{m}}, \bar{r}^*, \bar{\dot{p}}^*]$ for $t \geq t_1 \geq t_0$. They represent the stationary solutions for l and c in the model of equations (31)–(38):

$$\begin{bmatrix} l(t_1) \\ c(t_1) \end{bmatrix} = \begin{bmatrix} -\lambda & \dfrac{(1-\alpha)\gamma}{\delta\alpha} - \lambda & -\left(\dfrac{(1-\alpha)\gamma}{\delta\alpha} - \lambda\right) \\ 0 & \dfrac{\gamma}{\delta\alpha} & \dfrac{-\gamma}{\delta\alpha} \end{bmatrix} \begin{bmatrix} \bar{\dot{m}} \\ \bar{r}^* \\ \bar{\dot{p}}^* \end{bmatrix}. \tag{42}$$

It is clear that (42) is a special case of the general linear boundary conditions given in (23), with $M = 0$ and $N^{-1}r$ equal to the right-hand side of (42). Note that under these assumptions $[l(t), c(t)] = [l(t_1), c(t_1)]$ for $t > t_1$ as well. This two point boundary value problem can be solved using the method of adjoints of Section 3.

5. CONCLUSION

The methods presented in this paper represent a unified framework for solving a large class of continuous time linear rational expectations model. They also point the way to computational, numerical solution algorithms capable of handling high-dimensional models. The engineering libraries are rich in two-point boundary value algorithms, including the method of adjoints of Section 3. A computational algorithm for the solution method of Section 2, with or without linear restrictions on the state variables at the initial date, is also available (Austin and Buiter [**1**]). Miller and Salmon [**17, 18**] have shown how the same algorithm can be used in optimal or time-consistent (linear-quadratic) policy design in rational expectations models with one or many controllers.

London School of Economics

Manuscript received March, 1982; revision received July, 1983.

REFERENCES

[1] Austin, G. A., and W. H. Buiter: "'Saddlepoint': A Programme for Solving Continuous Time Linear Rational Expectations Models," University of Bristol Discussion Paper No. 82/132, November, 1982.

[2] Begg, D. K. H.: *The Rational Expectations Revolution in Macroeconomics: Theories and Evidence*. Oxford: Philip Allan, 1982.

[3] BEGG, D. K. H., AND M. B. HAGUE: "Financing Government Expenditure: Monetary and Fiscal Policy and Open Market Operations," LBS, Econometric Forecasting Unit Discussion Paper No. 104, December, 1982.

[4] BLANCHARD, O. J.: "The Monetary Mechanism in the Light of Rational Expectations," in *Rational Expectations and Economic Policy*, ed. by S. Fischer. Chicago: University of Chicago Press, 1980, pp. 75–116.

[5] BLANCHARD, O. J., AND C. M. KAHN: "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(1980), 1305–1311.

[6] BROCK, WILLIAM A.: "A Simple Perfect Foresight Monetary Model," *Journal of Monetary Economics*, 1(1975), 133–150.

[7] BUITER, W. H., AND M. MILLER: "Monetary Policy and International Competitiveness: The Problems of Adjustment," *Oxford Economic Papers*, (1981), 143–175.

[8] ———: "Real Exchange Rate Overshooting and the Output Cost of Bringing Down Inflation," *European Economic Review*, 18(1982), 85–123.

[9] ———: "Real Exchange Rate Overshooting and the Output Cost of Bringing Down Inflation: Some Further Results," London School of Economics, Centre for Labour Economics Discussion Paper No. 115, and in *Exchange Rates and International Macroeconomics*, ed. by J. Frenkel. Chicago: Chicago University Press, 1984.

[10] BUITER, W. H.: "Predetermined and Non-Predetermined Variables in Rational Expectations Models," *Economics Letters*, 10(1982), 49–54.

[11] CALVO, G. A.: "The Stability of Models of Money and Perfect Foresight: A Comment," *Econometrica*, 45(1977), 1737–1939.

[12] DIXIT, A.: "A Solution Technique for Rational Expectations Models with Applications to Exchange Rate and Interest Rate Determination," Mimeo, University of Warwick, November, 1980.

[13] DORNBUSCH, R.: "Expectations and Exchange Rate Dynamics," *Journal of Political Economy*, 84(1976), 1161–1176.

[14] DORNBUSCH, R., AND S. FISCHER: "Exchange Rates and the Current Account," *American Economic Review*, 70(1980), 960–971.

[15] GRAY, M., AND S. J. TURNOVSKY: "Expectational Consistency, Informational Lags, and the Formulation of Expectations in Continuous Time," *Econometrica*, 47(1979), 1457–1474.

[16] KRUGMAN, P.: "A Model of Balance of Payments Crises," *Journal of Money, Credit and Banking*, 11(1979), 311–325.

[17] MILLER, M., AND M. SALMON: "Optimal Control and Economic Dynamics: An Investigation of Co-ordinated and Decentralized Policy Formation in A Multi-Country Setting with Flexible Exchange Rates," Mimeo, University of Warwick, 1982.

[18] ———: "Dynamic Games and the Time Inconsistency of Optimal Policy in Open Economies," Mimeo, University of Warwick, June, 1983.

[19] MINFORD, A. P. L.: "Terminal Conditions, Uniqueness and the Solution of Rational Expectations Models," unpublished manuscript, University of Liverpool, 1979.

[20] OBSTFELD, M.: "Relative Prices, Employment and the Exchange Rate in an Economy with Foresight," *Econometrica*, 50(1982), 1219–1242.

[21] ROBERTS, S. M., AND J. S. SHIPMAN: *Two-Point Boundary Value Problems: Shooting Methods*. New York: Elsevier, 1972.

[22] SARGENT, T., AND N. WALLACE: "The Stability of Models of Money and Growth with Perfect Foresight," *Econometrica*, 41(1973), 1043–1048.

[23] TAYLOR, J. B.: "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations," *Econometrica*, 45(1977), 1377–1385.

[24] TURNOVSKY, S. J., AND E. BURMEISTER: "Perfect Foresight, Expectational Consistency, and Macroeconomic Equilibrium," *Journal of Political Economy*, 85(1977), 379–393.

[25] WILSON, C.: "Anticipated Shocks and Exchange Rate Dynamics," *Journal of Political Economy*, 87(1979), 636–647.