Deflationary Bubbles[1]

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Abstract

In an attempt to clean up an unruly literature, we specify the necessary and sufficient conditions for household optimality in a model where money is the only financial asset and provide the relevant proofs. We use our results to analyse when deflationary bubbles can and cannot exist. Our findings are in contrast to the results in several prominent contributions to the literature. We argue for particular specifications of the no-Ponzi-game restrictions on the household's and government's intertemporal budget constraints in a model with money and bonds. Using the restriction on the household we derive the necessary and sufficient conditions for household optimality. The resulting equilibrium terminal conditions are then used to demonstrate that the existence of bonds does not affect when deflationary bubbles can and cannot occur. This result differs from that in other recent works.

1 Introduction

This paper revisits the existence of deflationary bubbles and the terminal conditions that rule them out.[1] A striking feature of the current and past macroeconomic literature on deflationary bubbles is the divergence of opinion over the correct specification of both the transversality condition in models where money is the only financial asset and the correct specification of the transversality and long-run solvency, or "no-Ponzi-game", conditions in models where there are both money and bonds. Given the extent of the disagreement and confusion in the literature and the recent resurgence of interest in deflationary bubbles, we believe that it is useful to provide the correct (in the case of money only) and what we believe are the most attractive (in the case of money and bonds) terminal conditions. We use these conditions to specify when (rational) deflationary bubbles can and cannot exist.

The literature we are extending goes back to two seminal papers by Brock [5], [6]. Brock analyzes a closed-economy model where households save and receive liquidity services from holding money. At the time his papers were written, the necessary conditions for household optimality in infinite-horizon models – even for the special case of bounded utility functions – were not widely known. Brock correctly stated that a necessary condition is that the consumer must be indifferent between permanently reducing his money holdings by one unit, and enjoying a one-period marginal increase in utility due to the increased consumption, and leaving his money holdings unchanged and enjoying the discounted present value of the marginal utility of that unit of money forever. Brock's mathematical formulation of this idea is equivalent to an expression which looks like a transversality condition, but which is, in general, neither necessary or sufficient for household optimality and this has resulted in confusion.

The mispecification of a terminal condition in decades-old papers would be of little consequence except that the important results on deflationary bubbles in Brock [5], [6]

[1] By terminal conditions we mean restrictions that apply in the limit as time goes to infinity. These include transversality and "no-Ponzi-game" conditions.

and Obstfeld and Rogoff [23] depend on the exact specification of the necessary and sufficient conditions for household optimality. As many developed economies have experienced deflation in recent years, the issue is now of relevance to both academics and policy makers. The continued mistreatment of transversality conditions in such recent and important textbooks as Azariadis [1] and Obstfeld and Rogoff [24] also deserves mention. In this paper we provide the correct specification of the transversality condition that, together with the Euler equation, comprise the necessary and sufficient conditions for household optimality in Brock's model. We provide (for completeness) a proof that these conditions are sufficient for optimality, and using the technique in Kamihigashi's [17] elegantly simple proof, we also provide a proof that, under certain assumptions, the transversality condition is necessary.

Consistent with the early papers, we assume that the money supply grows at a constant rate $\mu > 1$ (or falls, if $\mu < 1$). Using the necessary and sufficient conditions for the household's optimisation problem, we then provide the correct specification of when deflationary bubbles can and cannot occur. We demonstrate that deflationary bubbles cannot occur when money growth is strictly positive ($\mu > 1$). We show, however, that when the money supply is contracting, but at a lower rate than the discount factor ($\beta < \mu < 1$), deflationary bubbles can occur; indeed, any separable utility function satisfying the usual regularity conditions can produce a deflationary bubble. We show that if the money supply contracts at a rate greater than the discount factor ($\mu \leq \beta$), then deflationary bubbles cannot exist.

Confusion about the correct terminal conditions also exists in models with both money and bonds. Turnovsky [28] (p. 24) and Ljungqvist and Sargent [18] (p. 511) assert that the household faces *two* transversality conditions: one for the terminal stock of bonds and one for the terminal stock of money. Perhaps more common, however, is the claim that there is only one transversality condition on the sum of the terminal stocks of debt and money: this assertion is made by Woodford [32] (p. 70) and Obstfeld and Rogoff [24] (p. 534). In addition to the household transversality conditions, in a model with money and debt, both the household and government each face another terminal condition in the form of a restriction on their feasible sets (or specification of their intertemporal budget constraint), often referred to as a "no-Ponzi-game" condition.[2] Here too there is dissent. Brock and Turnovsky [7] (p. 182) and McCallum [21] (p. 19) claim that households face a restriction on their terminal stock of non-monetary wealth. Farmer [12] (p. 236) and Benhabib, Schmitt-Grohé and Uribe [2] (p. 541), [3] (p. 73), [4] (p. 3) and Weil [29] (p. 39) on the other hand, assert that the restriction should be on the sum of the terminal stocks of monetary and non-monetary wealth. Buiter [8] (Section 2) argues that the government's terminal condition is a restriction on its terminal stock of bonds, while Canzoneri, Cumby and Diba [10] (p. 1224) state that the restriction should be on the sum of the government's terminal stocks of money and bonds.

We argue that the restriction on the household's and government's feasible sets is most appropriately a restriction on their terminal stocks of bonds. Given this assumption we demonstrate that the household has a single transversality condition that, along with the Euler equations, is necessary and sufficient for optimality. This condition says that the inner product of the vector of state variables (money and bonds) and the vector of present discounted values of marginal returns from increases in current state variables remains non-positive as time goes to infinity. Together with the no-Ponzi-game condition restricting the terminal stock of bonds, the single transversality condition is equivalent to two transversality conditions: one on money and one on bonds.

Using these two transversality conditions, we demonstrate that deflationary bubbles exist or fail to exist under the same circumstances as in the model with money only and we show that deflationary bubbles are characterised by nominal interest rates tending to zero. Given the two transversality conditions, the deflationary bubbles accompanied by strictly positive money growth in Woodford [32] and Benhabib, Schmitt-Grohé and Uribe [2] cannot exist.

Section 2 of the paper contains the model with money only; section 3 analyses the

[2] It is not uncommon for authors to make no distinction between these two types of restrictions.

existence of deflationary bubbles in the model of section 2; section 3 extends the model of section 2 to one with money and bonds. Section 4 is the conclusion.

2 The Model when Money is the Only Financial Instrument

2.1 The households

The economy is inhabited by a representative household and its government. Each period, the household receives an exogenous endowment of the single perishable consumption good and pays a lump-sum tax. It consumes the good and saves non-interest-bearing money. The household receives liquidity services from its money holdings and has preferences defined over paths of consumption and holdings of real balances represented by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, M_t^d/P_t),\ 0 < \beta < 1, \tag{1}$$

where $c_t \geq 0$ is time-t consumption, $M_t^d \geq 0$ is the household's time-t demand for nominal money balances, P_t is the period-t money price of the good and $u : \mathbb{R}_+^2 \rightarrow \mathbb{R} \cup \{-\infty\}$.[3]

The household maximises its utility subject to the sequence of within-period budget constraints

$$M_t^d/P_t = y - \tau_t - c_t + M_{t-1}^d/P_t,\ t \in \mathbb{Z}_+, \tag{2}$$

where $y > 0$ is the constant per-period endowment and $\tau_t < y + M_{t-1}^d/P_t$ is the period-t real lump-sum tax. Households take as given their initial money holdings $M_{-1}^d > 0$. We only consider outcomes where $1/P_t \in \mathbb{R}_{++}^2$ for every $t \in \mathbb{Z}_+$. There is, however, always a non-monetary equilibrium where $1/P_t = 0$ for every $t \in \mathbb{Z}_+$. In this outcome, money is not held and the household consumes its after-tax endowment each period.

Definition 1 *A sequence* $\left\{c_t, M_t^d/P_t\right\}$ *is said to be* ***feasible*** *if it satisfies (2). A feasible sequence* $\left\{c_t^*, M_t^{d*}/P_t\right\}$ *is said to be* ***optimal*** *if* $\liminf_{T\rightarrow\infty} \sum_{t=0}^{T} \beta^t [u\left(c_t, M_t^d/P_t\right)$ - $u\left(c_t^*, M_t^{d*}/P_t\right)] \leq 0$ *for every feasible sequence* $\left\{c_t, M_t^d/P_t\right\}$.[4]

[3] $\mathbb{R} \equiv (-\infty, \infty)$, $\mathbb{R}_+ \equiv [0, \infty)$, $\mathbb{R}_{++} \equiv (0, \infty)$ and $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.

[4] We use the notational convention $\{x_t\} \equiv \{x_t\}_{t=0}^{\infty}$.

We use the following assumptions:

Assumption 1 *(i)* $u(c,m) > -\infty$ *for* $(c,m) \in \mathbb{R}^2_{++}$*; (ii)* u *is* $\mathbb{C}^1$ *on* $\mathbb{R}^2_{++}$*, concave and has* $u_c(c,m) > 0$ *and* $u_m(c,m) \geq 0$ *for* $(c,m) \in \mathbb{R}^2_{++}$*.*

Assumption 2 *There exists a constant* $\rho \in \mathbb{R}$ *and a summable sequence* $\{b_t\}$ *such that* $\beta^t[u_c(c,m)\,c + u_m(c,m)\,m] \leq \rho\beta^t u(c,m) + b_t$*, for every* $(c,m) \in \mathbb{R}^2_{++}$*,* $t \in \mathbb{Z}_+$*.*[5]

Assumption 3 $u_c(c,m) \to \infty$ *as* $c \searrow 0$*,* $u_m(c,m) - u_c(c,m) \to \infty$ *as* $m \searrow 0$*.*

Assumption 4 *There exists* $\bar{u} \in \mathbb{R}_{++}$ *such that* $\lim_{m\to\infty} u_c(c,m) = \bar{u}$*.*

Assumption 5 *Either (i)* $u_m(c,m) > 0$ *for* $(c,m) \in \mathbb{R}^2_{++}$ *and* $\lim_{m\to\infty} u_m(c,m) = 0$ *or (ii) for every* $c \in \mathbb{R}_{++}$ *there exists* $\hat{m}(c) \in \mathbb{R}_{++}$ *such that* $u_m(c,m) > (=)\ 0$ *if* $m < (\geq)\ \hat{m}(c)$*.*

Not all of these assumptions are used for all of our results.

Proposition 1 *Assume that Assumption 1 holds. Sufficient conditions for the feasible sequence* $\left\{c_t^*, M_t^{d*}/P_t\right\}$*, where* $\left(c_t^*, M_t^{d*}/P_t\right) \in \mathbb{R}^2_{++}$*,* $t \in \mathbb{Z}_+$*, to be optimal are the Euler equation*

$$u_c(c_t, M_t^d/P_t) = u_m(c_t, M_t^d/P_t) + (\beta P_t/P_{t+1})\, u_c(c_{t+1}, M_{t+1}^d/P_{t+1}), \quad t \in \mathbb{Z}_+ \tag{3}$$

and the transversality condition

$$\lim_{t\to\infty} \beta^t \left[u_c(c_t, M_t^d/P_t) - u_m(c_t, M_t^d/P_t)\right] M_t^d/P_t \leq 0. \tag{4}$$

The proof of the above proposition for similar problems is standard (see, for example, Brock [5]). For completeness, a proof for this particular problem is provided in the Appendix.

Equation (3) is typical of the Euler equations that characterise investment in a consumer durable and has the following interpretation. The household is indifferent between a (small) one-unit increase in period-t consumption, which yields utility of $u_c\left(c_t, M_t^d/P_t\right)$, and foregoing this consumption and acquiring a one-unit increase in period-t real balances, which yield current utility of $u_m\left(c_t, M_t^d/P_t\right)$, and which can be traded next pe-

[5] A sequence $\{x_t\}$ is said to be summable if $\sum_{t=0}^{\infty} |x_t| < \infty$. Assumption 2 puts a limit on how fast utility can go to minus infinity when consumption or real balances go to zero. Suppose that $u(c,m) = h(c) + v(m)$. If $h(c) = \ln(c)$ or $c^{1-\theta}/(1-\theta)$ and $v(m) = \ln(m)$ or $m^{1-\theta}/(1-\theta)$, $\theta > 1$, then Assumption 2 is satisfied. However, if $h(c) = e^{-1/c}$ or $v(m) = e^{-1/m}$, then it is not satisfied. See Ekeland and Scheinkman [11].

riod for P_t/P_{t+1} units of the consumption good, which yields a discounted utility of $(\beta P_t/P_{t+1})\, u_c(c_{t+1}, M^d_{t+1}/P_{t+1})$.

The transversality condition in an infinite-horizon problem is often viewed as the analogue of the period-T complementary slackness condition in a T-period finite-horizon problem. This complementary slackness condition states that either $\beta^T[u_c(c_T, M^d_T/P_T) - u_m(c_T, M^d_T/P_T)] = 0$ or $M^d_T/P_T = 0$. If Assumption 3 (the Inada conditions at zero) holds then $M^d_T/P_T > 0$ and households are willing to hold real balances only up to the point where the marginal utility gain from the current liquidity services of money equals the marginal utility loss from decreased current consumption. In our infinite-horizon problem equation (4) implies that either the optimal value of the state variable, M^d_t/P_t, goes to zero as time goes to infinity or that its marginal contribution to the optimised value of the objective function, $\beta^t \left[u_c(c_t, M^d_t/P_t) - u_m(c_t, M^d_t/P_t)\right]$ becomes non-positive.

That the transversality condition is a necessary condition in problems similar to this one was first proved by Weitzman [30]. His proof, however, requires the strong assumption that the utility function is bounded and does not cover common utility functions such as $u\,(c) = \ln\,(c)$ or $c^{1-\theta}/\,(1-\theta)$, $\theta > 1$, where $u\,(c) \to -\infty$ when $c \searrow 0$. Ekeland and Scheinkman [11] showed that under certain assumptions, the transversality condition is also necessary for unbounded utility functions. Kamihigashi [17] relaxes some of Ekeland and Scheinkman's conditions and demonstrates that if utility does not go to minus infinity too quickly as consumption falls to zero and if the sequence of within-period discounted utilities is summable at an optimum, then the transversality condition must hold at that optimum.

Proposition 2 *Assume that Assumptions 1 and 2 hold. If the feasible sequence* $\left\{c^*_t, M^{d*}_t/P_t\right\}$*, where* $\left(c^*_t, M^{d*}_t/P_t\right) \in \mathbb{R}^2_{++}$*,* $t \in \mathbb{Z}_+$*, is optimal and if* $\left\{\beta^t u\left(c^*_t, M^{d*}_t/P_t\right)\right\}$ *is summable, then* $\left\{c^*_t, M^{d*}_t/P_t\right\}$ *satisfies the transversality condition (4).*

As the problem here is not identical to the one considered by Kamihigashi, a proof – which follows Kamihigashi's closely – is provided in the Appendix. Kamihigashi only requires that the set of points at which the utility function takes on a value strictly greater than minus infinity be an open set. Our stronger assumption that the utility function is strictly greater than negative infinity on $\mathbb{R}^2_{++}$ simplifies the proof.

The strategy of the proof is to compare an optimal sequence $\left\{c_t^*, M_t^{d*}/P_t\right\}$ with the following feasible perturbation: at time T, the household reduces its real balances to $\lambda M_T^{d*}/P_T$ and increases its consumption to $c_T^* + (1-\lambda)\, M_T^{d*}/P_T$, $0 \leq \lambda < 1$. Thereafter, its consumption and real balances are given by $\left\{\lambda c_t^*, \lambda M_t^{d*}/P_t\right\}_{t=T+1}^{\infty}$. Then, optimality requires that utility with the optimal sequence is at least as great as utility with the perturbation and this implies that $\beta^T[u\left(c_T^* + (1-\lambda)\, M_T^{d*}/P_T, \lambda M_T^{d*}/P_T\right)$ - $u\left(c_T^*, M_T^{d*}/P_T\right)]/\left(1-\lambda\right) \leq \sum_{t=T+1}^{s} \beta^t \left[u\left(c_t^*, M_t^{d*}/P_t\right)\right.$ - $\left.u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right)\right]/\left(1-\lambda\right)$. If the right-hand-side of this inequality can be shown to go to zero as $T \to \infty$, then applying $\lambda \to 1$ to the left-hand side, using the definition of a derivative and letting $T \to \infty$ establishes the result.

It is typical to consider models where Assumption 3 holds. In this case, the Euler equation (3) is also necessary and $\left(c_t^*, M_t^{d*}/P_t\right) \in \mathbb{R}^2_{++}$, $t \in \mathbb{Z}_+$. A proof of this can be found in Brock [5]. Then, by (3) and (4), the transversality condition can be written as:

$$\lim_{t\to\infty} \beta^t u_c(c_t, M_t^d/P_t) M_t^d/P_t = 0. \tag{5}$$

In the remainder of Section 2 and in Section 3, we assume that Assumptions 1 - 5 hold and we refer to equation (5) as the *transversality condition.*

2.2 The government

The government's within-period budget constraint, assumed to hold with equality, is

$$M_t/P_t = g - \tau_t + M_{t-1}/P_t,\ t \in \mathbb{Z}_+, \tag{6}$$

where $g \in [0, y)$ is constant per-period public spending and M_t is the time-t money supply. We assume a constant proportional growth rate for the money stock:

$$M_{t+1}/M_t = \mu > 0, \ t \in \mathbb{Z}_+. \tag{7}$$

The sequence of lump-sum taxes is endogenously determined to make public spending and the growth rate of the money stock consistent with the sequence of within-period government budget constraints and $\tau_t < y + M_{t-1}/P_t$; the assumption $g < y$ ensures this always possible.

2.3 Equilibrium

In equilibrium, $M_t^d = M_t, \ t \in \mathbb{Z}_+$ and

$$c_t = c \equiv y - g, \ t \in \mathbb{Z}_+. \tag{8}$$

Definition 2. *Given $\{M_t\}$, an **equilibrium** is a sequence of prices $\{P_t\}$ such that $P_t \in \mathbb{R}_{++}$, $t \in \mathbb{Z}_+$, and $\{c, M_t/P_t\}$ is optimal for the household.*

Definition 3. *If $\{P_t\}$ is an equilibrium sequence of prices then $\{m_t\}$, where $m_t \equiv M_t/P_t, \ t \in \mathbb{Z}_+$, is an **equilibrium sequence of real balances**.*

Substituting (8) and the money market clearing condition into (3) and (5) yields

$$\beta u_c(c, m_{t+1}) m_{t+1} = \mu [u_c(c, m_t) - u_m(c, m_t)] m_t, \ t \in \mathbb{Z}_+ \tag{9}$$

$$\lim_{t \to \infty} \beta^t u_c(c, m_t) m_t = 0. \tag{10}$$

In what follows we will use the following:

Definition 4 *A sequence of real balances $\{m_t\}$ is said to **satisfy the summability condition** if $\left\{\beta^t u(c, m_t)\right\}$ is summable.*

Propositions 1 and 2 and Definitions 2 - 4 yield the following remark.

Remark 1. A *sequence $\{m_t\}$, $m_t > 0$, $t \in \mathbb{Z}_+$, satisfying (9) and (10) is an equilib-*

rium sequence of real balances. If $\{m_t\}$ is an equilibrium sequence of real balances then it satisfies (9) and, if it satisfies the summability condition, then it satisfies (10).

There are two potential types of equilibria. First, given our constant fundamentals (y, g, μ), there is a *fundamental* (or Markov or "minimal-state-variable") equilibrium where $m_t = m > 0$ for every $t \in \mathbb{Z}_+$. Constant real balances clearly satisfy (10). By (9) such an equilibrium has

$$\mu u_m(c, m) - (\mu - \beta) u_c(c, m) = 0. \tag{11}$$

If $\mu < \beta$ or if $\mu = \beta$ and $u_m(c, m) > 0$, $m \in \mathbb{R}_{++}$, then the left-hand side of equation (11) is strictly positive for every $m \in \mathbb{R}_{++}$ and no fundamental equilibrium exists. If $\mu = \beta$ and there is satiation in real balances then any $m \geq \hat{m}(c)$ satisfies equation (11), where $\hat{m}(c)$ is as defined in Assumption 5(ii). Such an outcome is a Friedman Optimal Quantity of Money (OQM) equilibrium, where the nominal stock of money declines proportionally at the rate of time preference and the household is satiated at a finite stock of real balances. If $\mu > \beta$, then by Assumptions 3 - 5, $\mu u_m(c, m) - (\mu - \beta) u_c(c, m) \to \infty$ as $m \searrow 0$ and $\mu u_m(c, m) - (\mu - \beta) u_c(c, m) \to -(\mu - \beta)\bar{u} < 0$ as $m \to \infty$. Thus at least one fundamental equilibrium exists. For this case, the additional restriction that real balances are a normal good at any fixed point would ensure that the fundamental equilibrium is unique.[6]

In addition to fundamental equilibria, there can be a variety of non-fundamental (or non-stationary) equilibria. (See Matsuyama [19] and Azariadis [1]). An equilibrium can be stable, with monotonic or cyclical convergence; it can be unstable, with either monotonic or cyclical divergence; there can be limit cycles and there can be chaotic behaviour. We are interested in equilibria where real balances go to infinity; such equilibria are called deflationary bubbles.

[6] If u is twice differentiable we can write this condition as: $u_c u_{mm} - u_m u_{cm} < 0$ at a fixed point.

3 Deflationary bubbles

In this section we consider the existence of deflationary bubbles.

3.1 The definition of a deflationary bubble

Economists have many different definitions of bubbles, depending on the scenario under consideration. Here we have equilibria which depend solely on the fundamentals (and, hence, are not time varying) and equilibria which depend on time as well as on the fundamentals. Of the equilibria which depend on time as well as on the fundamentals, we will define the ones that go to infinity over time to be deflationary bubbles. This is a standard definition; see, for example, Obstfeld and Rogoff [23].

Definition 5. *A **deflationary bubble** is an equilibrium where $m_t \to \infty$ as $t \to \infty$.*

Note that this definition does not imply that an equilibrium sequence of prices which goes to zero must be a deflationary bubble or that all deflationary bubbles must have the price level going to zero. When the nominal money stock is falling, then a fundamental equilibrium has $P_{t+1}/P_t = M_{t+1}/M_t = \mu < 1$ and the price level goes to zero over time. When the nominal money stock is rising, a deflationary bubble has $P_{t+1}/P_t = \mu m_t/m_{t+1}$ and can be associated with rising prices if real balances are rising at a rate less than μ. Along such a path however, inflation will be less than the associated fundamental equilibrium's inflation rate of μ.

3.2 Brock's restriction on optimal programmes

Writing before the publication of Weitzman's [30] proof of the necessity of the transversality condition for bounded utility functions, Brock [6] (p. 140) proposed a necessary condition for optimal programmes. He made a "no-arbitrage" argument that at an optimum, the household must be indifferent between permanently reducing his real balances by one unit today and enjoying a marginal increase in today's utility due to higher consumption and leaving his real balances unchanged and enjoying the discounted utility from the services of that unit of money forever. Brock expressed this condition mathematically as[7]

$$u_c(c_t, M_t^d/P_t) = \sum_{s=0}^{\infty} (\beta^s P_t/P_{t+s})\, u_m(c_{t+s}, M_{t+s}^d/P_{t+s}). \tag{12}$$

As shown in the previous section, there are two necessary conditions for household optimality: the first is the Euler equation, which relates time-t variables to time-$t+1$ variables. The necessity of this condition is shown by switching small amounts of consumption and real balances between time-t and time-$t+1$ and then demonstrating that the first path yields at least as high utility as the second. The second type is the transversality condition which is a condition on the asymptotic behaviour of consumption and real balances as time goes to infinity.[8] Brock's proposed perturbation is a change in *current* consumption and real balances and, hence, does not establish a transversality condition. Indeed, Brock [6] shows that the transversality condition (4) is a sufficient condition and this suggests that he did not view equation (12) as a transversality condition.

Solving the Euler equation (3) forward yields

$$u_c\left(c_t, M_t^d/P_t\right) = \sum_{s=0}^{\infty} \frac{\beta^s P_t u_m(c_{t+s}, M_{t+s}^d/P_{t+s})}{P_{t+s}} + \lim_{T\to\infty} \frac{\beta^T P_t u_c(c_{t+T}, M_{t+T}^d/P_{t+T})}{P_{t+T}}. \tag{13}$$

By equation (13), equation (12) is equivalent to

$$\lim_{T\to\infty} \beta^T u_c(c_{t+T}, M_{t+T}^d/P_{t+T})\,(1/P_{t+T}) = 0. \tag{14}$$

At an equilibrium this can be written as

$$\lim_{t\to\infty} (\beta/\mu)^t\, u_c(c, m_t) m_t = 0. \tag{15}$$

Thus, Brock's mathematical formulation of his "no-arbitrage" argument, when combined

[7]Brock assumed a separable utility function; this condition is the non-separable analogue to his condition.

[8]See Ekeland and Scheinkman [11] for a discussion of this.

with the Euler equation produces an equation that *looks* like a transversality condition and this has apparently led to a substantial amount of confusion. Obstfeld and Rogoff [22] (p. 681), [23] (p. 360-1) and Gray [15] (p. 110) and – more recently – Azariadis [1] (p. 403,405) and Obstfeld and Rogoff [24] (p. 541-542) all reproduce Brock's "no-arbitrage" argument and use it to claim that equation (14) is a transversality condition and necessary for household optimality.[9] In the rest of the paper we refer to condition (14) as the *GABOR* (Gray-Azariadis-Brock-Obstfeld-Rogoff) *condition*.

The proof of Proposition 2, demonstrating that transversality condition (4) is necessary for household optimality, employs Brock's proposed perturbation of current (that is time-t) and future real balances and consumption. However, as seen in equation (33) in the Appendix (and also in the discussion under the statement of Proposition 2 in the text), the mathematical expression for this perturbation differs from Brocks, and is used only asymptotically – as time goes to infinity.

The *GABOR* condition has been used to study the theoretical existence of deflationary bubbles by Brock [5], [6] and Obstfeld and Rogoff [23] in their well-known papers. In his Theorem 3 (p. 140), Brock [6] assumes a separable utility function: $u(c,m) = h(c) + v(m)$. He attempts to show that for $\mu > \beta$ no deflationary bubble can satisfy the *GABOR* condition. As this is not in general true, he imposes an additional condition: there exists a $\lambda < 0$ such that for sufficiently large m, $v'(m) < m^{\lambda}$. This condition is weak, if not particularly intuitive. Thus, if equilibria must satisfy the *GABOR* condition, then it is only in "pathological" cases that deflationary bubbles can exist. Obstfeld and Rogoff [23] consider the case of $\mu > 1$ and show that under the stricter, but more intuitively appealing, condition that utility is bounded above in real balances, imposing the GABOR condition is sufficient to rule out deflationary bubbles.[10]

[9] Gray [15] notes that transversality conditions generally require the product of the state variable and its discounted value to go to zero as time goes to infinity, as in equation (4). In her paper and in Obstfeld and Rogoff [24] the money stock is constant and hence equation (4) and equation (14) turn out to be the same. However, the technique they use to derive (14) would yield (4) if the money supply were not constant.

[10] Obstfeld and Rogoff [23] restrict attention to this case because they claim that Brock's proposed perturbation of an optimal sequence is not feasible otherwise. Equations (2) and (33) (in the Appendix)

By (10) and (15) the transversality condition implies the *GABOR* condition when $\mu > 1$ and the *GABOR* condition implies the transversality condition when $\mu < 1$; the conditions are equivalent when $\mu = 1$. Thus, if $\mu > 1$ and a sequence $\{m_t\}$ satisfies the summability condition and has $m_t \to \infty$ as $t \to \infty$, then using the *GABOR* condition to rule out this candidate deflationary bubble is legitimate: the transversality condition is necessary for household optimisation (Proposition 2) and the *GABOR* condition is necessary for the transversality condition. But, it makes more sense to use the stronger transversality condition. In the next subsection we present a simple proof ruling out deflationary bubbles that requires no additional assumptions.

If $\mu > 1$, $m_t \to \infty$ and $\{m_t\}$ does *not* satisfy the summability condition, then neither the transversality condition nor the *GABOR* condition have been demonstrated to be necessary. Hence, they cannot be used to rule out deflationary bubbles.[11]

When $\mu < 1$ it is not legitimate to use the *GABOR* condition to rule out deflationary bubbles satisfying (9) and (10). As the transversality condition is sufficient (Proposition 1), the stronger (in this case) *GABOR* condition cannot be necessary. In the next subsection we show that any sequence $\{m_t\}$ satisfying (9) and where $m_t \to \infty$ also satisfies the transversality condition and is an equilibrium deflationary bubble.

3.3 The relationship between the transversality condition and the "no-bubble" boundary condition

Turning briefly to a different scenario, consider the market for a particular company's stock in a model without money in the utility function. Under certainty the household's Euler equation corresponding to that stock says that $p_t u'(c_t) = \beta (p_{t+1} + d_{t+1}) u'(c_{t+1})$, $0 < \beta < 1$,where u is the within-period utility function and c_t, p_t and d_t are the time-t consumption demand, stock price (in terms of the consumption good) and (exogenous)

make it clear that the perturbation – as specified in (33) – is always feasible.

[11] Suppose that $u(c, m) = h(c) + v(m)$, where $v(m) = m^{1-\theta}/(1-\theta)$ if $1 \neq \theta > 0$ and $v(m) = \ln(m)$ if $\theta = 1$. If $\{m_t\}$ satisfies (9) and $m_t \to \infty$ as $t \to \infty$ then $\{m_t\}$ satisfies the summability condition if $\beta^\theta \mu^{1-\theta} < 1$. Details on request.

dividend, respectively. Suppose that, as in our model, $c_t = c \equiv y - g$, $t \in \mathbb{Z}_+$. Then, solving the Euler equation forward would yield $p_t = \sum_{s=1}^{\infty} \beta^s d_{t+1} + \lim_{T\to\infty} \beta^T p_{t+T}$. Thus, the stock price consists of a term $F_t \equiv \sum_{s=1}^{\infty} \beta^s d_{t+1}$, which depends on the fundamentals (that is, the dividends), and a term $C_t \equiv \lim_{T\to\infty} \beta^T p_{t+T}$.

This latter term may be strictly positive if investors have self-fulfilling expectations that the price will rise by more than is justified by the fundamentals. Alternatively, this term may be written as $C_t = k/\beta^t$, where $k \geq 0$. Solutions where $k > 0$ and, hence $C_t \neq 0$ are often referred to as *rational* or *equilibrium bubbles.* They might be viewed as unlikely or not "sensible" as they are not Markov or "minimal-state-variable" solutions in McCallum's [20] sense as they depend on an extraneous variable: calendar time. In theoretical models it is typical to impose the boundary condtion $\lim_{T\to\infty} \beta^T p_{t+T} = 0$ to rule out such equilibria. In empirical models, deviations between p_t and the fundamental component, F_t, are often referred to as a *bubble* and researchers often test for the existence of a bubble by testing whether the price can be explained by the fundamentals: in this example, this would be testing whether $p_t = F_t$.

The boundary condition ruling out bubble equilibria *looks* like a transversality condition and some researchers, for example Froot and Obstfeld [14], call this condition a transversality condition. However, it is not related to the transversality condition which, under certain assumptions, is necessary and sufficient for household optimality. In the model of stock prices this transversality condtion would be $\lim_{T\to\infty} \beta^T u'(c_T)\, p_T s_T \leq 0$, where s_t is the household's time-t holdings of the stock. In the model of this paper, when the Euler equation (3) is solved forward to find Brock's condition (13), it *looks* similar to the procedure where the equilibrium condition $p_t u'(c_t) = \beta (p_{t+1} + d_{t+1}) u'(c_{t+1})$, $c_t = c, t \in \mathbb{Z}_+$ is solved forward to find the stock price as the sum of a fundamental solution (F_t) and a bubble component (C_t) and the bubble component is then set equal to zero.

In our model, the analogous procedure for decomposing the general form for an equilibrium into fundamental and bubble components is not to solve the household's Euler equation forward, but to solve the *equilibrium condition* (9) forward. Equation (9) is not, in general, linear and this prevents a closed-form solution, but in this model with constant fundamentals, the analogue to F_t is the constant fundamental equilibrium $\bar{m}$ that solves (11). For the particular case of $u(c,m) = h(c) + \ln m$, equation (9) is linear and can be solved forward to find $m_t = \bar{m} + \lim_{T\to\infty} (\beta/\mu)^T m_{t+T}$. Thus, for this special case the GABOR condition can be used to rule out paths of real balances which are consistent with household optimisation and market clearing, but which depend on a variable other than the fundamentals.

3.4 The existence of deflationary bubbles

In this subsection we use the equilibrium conditions to characterise when deflationary bubbles can and cannot exist.

Proposition 3. *Suppose that a sequence* $\{m_t\}$ *has* $m_t \to \infty$ *as* $t \to \infty$. (i) If $\mu > 1$ *and* $\{m_t\}$ *satisfies the summability condition, then* $\{m_t\}$ *is not an equilibrium sequence of real balances. (ii) If* $\beta < \mu < 1$ and $\{m_t\}$ *satisfies (9) then* $\{m_t\}$ *is an equilibrium sequence of real balances.*

Proof. Suppose that $\mu > 1$ and let $x_t \equiv u_c(c, m_t)m_t > 0$. By (9), $x_{t+1}/x_t = (\mu/\beta)[1 - u_m(c,m_t)/u_c(c, m_t)]$, $t \in \mathbb{Z}_+$. By Assumptions 3 and 4, $x_{t+1}/x_t \to \mu/\beta$ as $m_t \to \infty$. Thus, $\forall\epsilon > 0, \exists T \in \mathbb{Z}_+$ such that $x_{T+t+1}/x_{T+t} > \mu/\beta - \epsilon$, $t \in \mathbb{Z}_+$. Let $\epsilon = (\mu - 1)/\beta$. Then $\beta^{T+t}x_{T+t} > \beta^T x_T > 0$, $t \in \mathbb{Z}_{++}$. Hence, $\beta^{T+t}x_{T+t}$ cannot go to zero as $t \to \infty$ and (10) is violated. This yields (i). If $\{m_t\}$ satisfies (9) then $x_{t+1}/x_t \leq \mu/\beta, t \in \mathbb{Z}_+$. Thus, $\beta^t x_{T+t} \leq \mu^t x_T \to 0$ as $t \to \infty$, $T \in \mathbb{Z}_+$. Thus (10) is satisfied.

When $\beta < \mu < 1$ it is easy to find examples of deflationary bubble equilibria; indeed, any separable utility function $u(c,m) = h(c) + v(m)$, where Assumptions 1 - 5 are satisfied, produces deflationary bubbles.[12] .

When $\mu = 1$, the transversality condition and the *GABOR* condition are identical and Brock's and Obstfeld and Rogoff's results apply here. Obstfeld and Rogoff [23] provide an example (suggested by Guillermo Calvo and Roque Fernandez) of a utility function where the *GABOR* condition (and hence the transversality condition) alone is insufficient

[12] When u is separable (9) implies that $dm_{t+1}/dm_t > m_{t+1}/m_t$. Thus, dm_{t+1}/dm_t is strictly greater than one at any steady state.

to rule out deflationary bubbles. This utility function is separable and has the property that the marginal utility of money is $1/\ln(m)$ for m large. If $m_{-1} > \bar{m}$, the sequence, $\{m_t\}$ that satisfies equation (9) also satisfies the *GABOR* condition and has $m_t \to \infty$.

We now consider the case of $\mu \leq \beta$. We show that when money growth equals the discount factor and there is satiation in real balances, deflationary bubbles cannot exist. When $\mu < \beta$ fundamental equilibria do not exist. This case is not considered by Brock [5], [6]. We show that there are no deflationary bubbles in this case either. Both results are a consequence of the Euler equation, rather than the transversality condition.

Proposition 4. *Suppose that a sequence $\{m_t\}$ has $m_t \to \infty$ as $t \to \infty$. If $\mu \leq \beta$ then $\{m_t\}$ cannot be an equilibrium sequence of real balances.*

Proof. By (9), $m_{t+1} = (\mu/\beta)\left[u_c(c, m_t)/u_c(c, m_{t+1}) - u_m(c, m_t)/u_c(c, m_{t+1})\right] m_t$, $t \in \mathbb{Z}_+$. Thus, by Assumption 1, $m_{t+1} \leq (\mu/\beta)\left[u_c(c, m_t)/u_c(c, m_{t+1})\right] m_t$, $t \in \mathbb{Z}_+$ and, hence, $m_t \leq (\mu/\beta)^t \left[u_c(c, m_0)/u_c(c, m_t)\right] m_0$. Thus, $\lim_{t\to\infty} m_t \leq \lim_{t\to\infty} (\mu/\beta)^t \left[u_c(c, m_0)/u_c(c, m_t)\right] m_0$ $\leq \left[u_c(c, m_0)/\bar{u}\right] m_0 < \infty$.

When $\mu = \beta$, Brock [6] shows that if $u\,(c, m_t) = h\,(c) + v\,(m)$, where $v'\,(m) > (<, =)$ 0 for $m < (>, =)$ 0 and $\lim_{m\to\infty} v(m) = -a < 0$, $a > 0$, then deflationary bubbles satisfy (9) and (10) if and only if a is sufficiently small.

In the final proposition in this subsection we demonstrate that using the *GABOR* condition rules out the OQM equilibrium.

Proposition 5. *The GABOR condition rules out Friedman's Optimal Quantity of Money equilibrium.*

Proof. Let $\mu = \beta$ and let $m_t = m' \geq \hat{m}$. Then $(\beta/\mu)^t u_c(c, m_t) m_t = u_c(c, m')m' > 0$ and the *GABOR* condition is not satisfied.

4 Deflationary Bubbles with Money and Government Bonds

In this section we extend the model to allow for government debt as well as money.

4.1 Households

We assume that the government issues nominal bonds, in addition to money.[13] Since the nominal interest rate on money is assumed to be zero, an equilibrium with valued bonds requires that the nominal interest rate be non-negative and to be strictly positive when the household is not satiated in real balances. We only consider outcomes where this is true.[14]

Denote the period-t household demand for bonds by B_t^d and let $a_t^d \equiv \left(M_t^d + B_t^d\right)/P_t$. The household's within-period budget constraint is

$$a_t^d = (1 + i_t)\,(P_{t-1}/P_t)\,a_{t-1}^d + y - \tau_t - c_t - (i_t P_{t-1}/P_t)\,M_{t-1}^d/P_{t-1},\ t \in \mathbb{Z}_+, \tag{16}$$

where i_t is the nominal interest rate between periods $t-1$ and t and $\tau_t < (1+i_t)\,(P_{t-1}/P_t)\,a_{t-1}^d$ $+\, y - (i_t P_{t-1}/P_t)\,M_{t-1}^d/P_{t-1}$, $t \in \mathbb{Z}_+$. We assume that the household's initial holdings of money and bonds, $M_{-1} > 0$ and B_{-1}, respectively, are given.

The household cannot run a Ponzi scheme where it borrows ever-increasing amounts to service its previously accumulated debt. We impose the restriction that the present discounted value of the household's terminal (non-monetary) debt must be non-negative:

$$\lim_{t\to\infty} B_t^d / \prod_{s=0}^t (1 + i_s) \geq 0. \tag{17}$$

Many recent papers, however, contain an alternative restriction:

$$\lim_{t\to\infty} \left(B_t^d + M_t^d\right) / \prod_{s=0}^t (1 + i_s) \geq 0. \tag{18}$$

This no-Ponzi-game condition is an assumption about how the world works and therefore a matter of opinion. To see why we prefer (17), imagine an analogous T-period model, where $T < \infty$. Typically, one would impose a restriction similar in spirit to (17): in the

[13] Including real bonds is trivial and adds to the notation without changing the results.

[14] In the money-and-bonds model too, we do not consider the non-monetary equilibrium with $P_t^{-1} = 0,\ t \geq 0$.

last period all previously paid accumulated debt must be repaid and no additional borrowing can take place. Suppose instead that one imposed a restriction similar to (18): outstanding debt need not be repaid if the household holds real balances equal to the outstanding debt. It is difficult to see why anyone would lend to the household when there is no future in which they would be repaid. In addition, if utility is strictly increasing in (end-of-period) real balances, households would want to hold an infinite amount of real balances and an infinite amount of debt in the last period and the household's optimisation problem would have no solution.

Similarly, in an infinite-horizon model it is difficult to see why any counterparty would want the present discounted value of its terminal debt to be strictly positive. If it is argued that there is some unusual circumstance where a counterparty – say, the government – is willing to lend ever increasing amounts to the private sector then using restriction (18) might be appropriate, but its use presents a problem. The conventional method of proving the sufficiency of the Euler and transversality conditions requires the use of the stronger condition (17).[15] Thus, we are uncertain what the sufficient conditions for household optimality are under the alternative restriction (18).

Definition 5 *A sequence* $\left\{c_t, M_t^d/P_t, a_t^d\right\}$ *is said to be* ***feasible*** *if (16) and (17) are satisfied. The definition of* ***optimality*** *is as in Definition 1.*

Proposition 6 *Assume that Assumption 1 holds. Sufficient conditions for the feasible sequence* $\left\{c_t^*, M_t^{d*}/P_t, a_t^{d*}\right\}$*, where* $\left(c_t^*, M_t^{d*}/P_t\right) \in \mathbb{R}_{++}^2$*,* $t \in \mathbb{Z}_+$*, to be optimal are that it satisfies the budget constraint* (16)*, the Euler equations*

$$\frac{u_m(c_t, M_t^d/P_t)}{u_c(c_t, M_t^d/P_t)} = \frac{i_{t+1}}{1+i_{t+1}},\ t \in \mathbb{Z}_+, \tag{19}$$

$$\beta(1+i_{t+1})\left(P_t/P_{t+1}\right) u_c\left(c_{t+1}, M_{t+1}^d/P_t\right) = u_c\left(c_t, M_t^d/P_t\right),\ t \in \mathbb{Z}_+, \tag{20}$$

and the transversality condition.

$$\lim_{t\to\infty}\left\{\beta^t u_c\left(c_t, M_t^d/P_t\right) B_t^d/P_t + \left[u_c\left(c_t, M_t^d/P_t\right) - u_m\left(c_t, M_t^d/P_t\right)\right] M_t^d/P_t\right\} \leq 0. \tag{21}$$

Proof. See the Appendix.

[15] This is seen in the proof of Proposition 6 in the Appendix.

The transverality condition (21) appears unusual, but has the same interpretation as the transversality condition for the multi-sector growth model in Stokey and Lucas [26]: the inner product of the vector of state variables and the vector of present discounted values of marginal returns from increases in current state variables is non-positive as time goes to infinity. Here the value of the marginal return of an increase in current bond holdings is the marginal utility loss due to foregone consumption; the value of a marginal increase in current money holdings is the marginal utility loss due to foregone consumption less the marginal utility gain due to increased liquidity services.

Proposition 7 *Assume that Assumptions 1 and 2 hold. If the feasible sequence* $\left\{c_t^*, M_t^{d*}/P_t, a_t^{d*}\right\}$, *where* $\left(c_t^*, M_t^{d*}/P_t\right) \in \mathbb{R}_{++}^2$, $t \in \mathbb{Z}_+$, *is optimal then it satisfies the transversality condition* (21).

Proof. See the Appendix.

In the remainder of this section we assume that Assumptions 1 - 5 hold. In this case the Euler condidtion is necessary as well. By (17) and (20), the no-Ponzi-Game condition can be rewritten as

$$\lim_{t\to\infty} \beta^t u_c\left(c_t, M_t^d/P_t\right) B_t^d/P_t \geq 0. \tag{22}$$

By (19) and non-negative nominal interest rates, $u_c\left(c_t, M_t^d/P_t\right) - u_m\left(c_t, M_t^d/P_t\right) > 0$. Thus, $\lim_{t\to\infty} \beta^t\left[u_c\left(c_t, M_t^d/P_t\right) - u_m\left(c_t, M_t^d/P_t\right)\right] M_t^d/P_t \geq 0$ and (21) and (22) together are equivalent to the pair of conditions

$$\lim_{t\to\infty} \beta^t u_c\left(c_t, M_t^d/P_t\right) B_t^d/P_t = 0, \;\; \lim_{t\to\infty} \beta^t\left[u_c\left(c_t, M_t^d/P_t\right) - u_m\left(c_t, M_t^d/P_t\right)\right] M_t^d/P_t = 0. \tag{23}$$

Substituting (19) into (23) implies $\lim_{t\to\infty} \beta^t u_c\left(c_t, M_t^d/P_t\right) M_t^d/P_t/\left(1+i_{t+1}\right) = 0$. If the interest rate does not go to infinity (which by (19) and Assumption 3 would require an *inflationary* bubble), this condition can be expressed in the more familiar form

$$\lim_{t\to\infty} \beta^t u_c\left(c_t, M_t^d/P_t\right) M_t^d/P_t = 0. \tag{24}$$

4.2 The government

Let $b_t \equiv B_t/P_t$, where B_{t-1} is the government's outstanding stock of bonds at the beginning of period t, and let $a_t \equiv m_t + b_t$. We restrict the government to rules satisfying $a_t + y - g > 0$, $t \in \mathbb{Z}_+$ amd assume that $a_{-1} + y - g > 0$. The government's period-t budget constraint is

$$a_t = (1 + i_t)\,(P_{t-1}/P_t)\,a_{t-1} + g - \tau_t - (i_t P_{t-1}/P_t)\,m_{t-1},\ t \in \mathbb{Z}_+. \tag{25}$$

It is typical to express the government's long-run solvency constraint as

$$\lim_{t\to\infty} (M_t + B_t) \,/\, \textstyle\prod_{s=0}^{t}(1 + i_s) \leq 0. \tag{26}$$

However, we assume that the money in the model is unbacked fiat money. Thus, as it is irredeemable, it is not a liability of the government (see Buiter [8], [9]) and the government's solvency constraint is

$$\lim_{t\to\infty} B_t / \textstyle\prod_{s=0}^{t}(1 + i_s) \leq 0. \tag{27}$$

We view the government as choosing $\{M_t, B_t\}$ such that, given prices and g, $\{B_t\}$ satisfies (27).[16] The sequence of taxes is then endogenously chosen to satisfy (25).

4.3 Market clearing

Market clearing requires that $m_t^d = m_t$ and $a_t^d = a_t$, $t \in \mathbb{Z}_+$. As before, the resource constraint implies that $c_t = c \equiv y - g$, $t \in \mathbb{Z}_+$. The assumption that $a_t + y - g > 0$ ensures that it is always possible to find a sequence of taxes satisfying the assumed

[16] There exists a substantial empirical literature testing whether or not governments satisfy (27). Using US data, Hamilton and Flavin [16] and Trehan and Walsh [27] find supportive evidence; Wilcox [31], who uses US data, and Smith and Zin [25], who use Canadian data, find evidence that governments follow unsustainable policies, suggesting either that the government is playing a Ponzi game or that a change in policies is expected Of course if governments do not satisfy (27), then they do not satisfy the the stronger condition (26).

restriction $\tau_t < (1+i_t)(P_{t-1}/P_t)\,a^d_{t-1} + y$ - $(i_t P_{t-1}/P_t)\,M^d_{t-1}/P_{t-1} = a_t + y - g + \tau_t$, $t \in \mathbb{Z}_+$. Thus by equations (19), (20) and (23) we have the following definition:

Definition 6 *An* ***equilibrium sequence of real balances*** *is a sequence* $\{m_t\}$ *such that* $m_t \in \mathbb{R}_{++}$, $t \in \mathbb{Z}_+$, *and*

$$\beta u_c(c, m_{t+1}) m_{t+1} = \mu[u_c(c, m_t) - u_m(c, m_t)] m_t,\ t \in \mathbb{Z}_+ \tag{28}$$

$$\lim_{t\to\infty} \beta^t u_c(c, m_t) b_t = \lim_{t\to\infty} \beta^t u_c(c, m_t) m_t = 0. \tag{29}$$

In an equilibrium, nominal interest rates are given by

$$i_{t+1} = \frac{u_m(c, m_t)}{u_c(c, m_t) - u_m(c, m_t)} > (=)\ 0 \text{ if } u_m(c, m_t) > (=)\ 0,\ t \in \mathbb{Z}_+ \tag{30}$$

As before, a fundamental equilibrium exists when $\mu > \beta$. By (28) and (30) it has the associated nominal interest rate $\bar{\imath} = (\mu - \beta)/\beta$.

4.4 Deflationary bubbles in a model with bonds

We demonstrate that adding government bonds to the model does not change the results of the previous section.

Proposition 8 *Suppose that* $\{b_t\}$ *satisfies (29) and that* $\{m_t\}$ *has* $m_t \to \infty$ *as* $t \to \infty$. (i) If $\mu > 1$ *and* $\{m_t\}$ *satisfies the summability condition, then* $\{m_t\}$ *is not an equilibrium sequence of real balances. (ii) If* $\beta < \mu < 1$ and $\{m_t\}$ *satisfies (28) then* $\{m_t\}$ *is an equilibrium sequence of real balances and* $i_{t+1} \to 0$. *(iii) If* $\mu \leq \beta$ *then* $\{m_t\}$ *is not an equilibrium sequence of real balances.*

Proof. The proof of Proposition 3 demonstrates that if $\{m_t\}$ satisfies (28) and has $m_t \to \infty$ as $t \to \infty$, then $\{m_t\}$ satisfies (29) when $\beta < \mu < 1$ and fails to satisfy (29) when $\mu > 1$. This yields (i) and (ii). The proof of Proposition 4 demonstrates that if $\mu \leq \beta$ then $\{m_t\}$ cannot satisfy (28). This yields (iii).

Our results are in contrast to the results in Woodford [32] (p. 131-135) and Benhabib, Schmitt-Grohé and Uribe [2] (section VI.A), who find that adding debt changes the regions of the parameter space where deflationary bubbles can exist. They use the alternative no-Ponzi-game condition (18). They then demonstrate that, when money is growing at a strictly positive rate, it is possible to have a sequence of real balances that tends to infinity and that satifies the Euler equations and this alternative no-Ponzi-game condition. This bubble has the property that, as the discounted present value of money balances goes to infinity, the present discounted value of government debt goes to minus infinity.

5 Conclusion

Terminal conditions have been problematic for monetary economists. Their specification differs from paper to paper and textbook to textbook, although the same model is employed. Restrictions on feasible sets (that is, the "no-Ponzi-game" conditions) are commonly not distinguished from the necessary and sufficient conditions for optimality, given the particular choice of a restriction on the feasible set. The intent of this paper is to provide a coherent treatment of the subject for two common models: a model with money in the utility function where money is the only financial asset and a model with money in the utility function and both money and bonds serving as financial assets.

We specify the necessary and sufficient conditions for household optimality, and we provide the relevant proofs. In the model with money only, we demonstrate that the transversality condition which is part of the necessary and sufficient conditions, differs from a condition employed elsewhere in the literature. In the model with money and bonds we argue for particular restrictions on the household's and government's feasible sets. Using the restriction on the household's feasible set, we find the household transversality condition that, together with the Euler equation, constitutes the necessary and sufficient conditions for household optimality. Our result implies that in equilibrium there are a pair of terminal conditions that must be satisfied – one on money and one on bonds – rather than the single condition on the sum of the stock of money and bonds that frequently appears.

The resurgence of actual and prospective disinflation in industrialised countries has resulted in new interest in the possibility of self-fulfilling deflationary expectations. We use our results to demonstrate that, whether there is only money or whether there are money and bonds, deflationary bubbles cannot occur with reasonable utilty functions and positive nominal money growth. However, if the nominal money stock is falling, but not faster than the discount factor, then any sensible separable utility function can produce a deflationary bubble. If households have satiation in money balances, then a decline in money growth that supports Friedman's optimal quantity of money equilibrium (that is, a decline equal to the discount factor) cannot produce deflationary bubbles.

6 Appendix

Proof of Proposition 1. Let $\{c_t, M_t^d/P_t\}$ be a feasible sequence. By (2)

$$\begin{aligned} D &\equiv \lim_{T\to\infty}\inf \textstyle\sum_{t=0}^{T} \beta^t \left[u\left(c_t, M_t^d/P_t\right) - u\left(c_t^*, M_t^{d*}/P_t\right)\right] \\ &= \lim_{T\to\infty}\inf \textstyle\sum_{t=0}^{T} \beta^t [u\left(y - \tau_t - M_t^d/P_t + M_{t-1}^d/P_t, M_t^d/P_t\right) \\ &\quad -u\left(y - \tau_t - M_t^{d*}/P_t + M_{t-1}^{d*}/P_t, M_t^{d*}/P_t\right)]. \end{aligned}$$

Then by Assumption 1,

$$\begin{aligned} D &\leq \lim_{T\to\infty} \textstyle\sum_{t=0}^{T} \beta^t [u_c\left(c_t^*, M_t^{d*}/P_t\right)\left(M_{t-1}^d/P_t - M_{t-1}^{d*}/P_t\right) + \\ &\quad -u_c\left(c_t^*, M_t^{d*}/P_t\right)\left(M_t^d/P_t - M_t^{d*}/P_t\right) + u_m\left(c_t^*, M_t^{d*}/P_t\right)\left(M_t^d/P_t - M_t^{d*}/P_t\right)] \\ &= \lim_{T\to\infty} \textstyle\sum_{t=-1}^{T-1} \beta^t \{u_c\left(c_{t+1}^*, M_{t+1}^{d*}/P_{t+1}\right)\left(M_t^d/P_t - M_t^{d*}/P_t\right)\beta P_t/P_{t+1} + \\ &\quad - \lim_{T\to\infty} \textstyle\sum_{t=0}^{T} \beta^t \left[u_c\left(c_t^*, M_t^{d*}/P_t\right) - u_m\left(c_t^*, M_t^{d*}/P_t\right)\right]\left(M_t^d/P_t - M_t^{d*}/P_t\right)\}. \end{aligned}$$

Thus, by (3) and the given initial conditions

$$D \leq -\lim_{T\to\infty} \beta^T \left[u_c\left(c_T^*, M_T^{d*}/P_T\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right)\right]\left(M_T^d/P_T - M_T^{d*}/P_T\right).$$

By (3) and Assumption 1, $u_c\left(c_T^*, M_T^{d*}/P_T\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right) > 0$; hence,

$$D \leq \lim_{T\to\infty} \beta^T \left[u_c\left(c_T^*, M_T^{d*}/P_T\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right)\right] M_T^{d*}/P_T. \tag{31}$$

Equation (4) implies that the right-hand-side is non-positive, establishing the result.

Proof of Proposition 2. The proof requires two lemmas.

Lemma 1. If there exists a constant $\lambda^* \in (0,1)$ and a summable sequence $\{e_t\}$ such that

$$\frac{\beta^t u\left(c_t^*, M_t^{d*}/P_t\right) - \beta^t u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right)}{1-\lambda} \leq e_t \ \forall \lambda \in [\lambda^*, 1), \forall t \in \mathbb{Z}_+, \tag{32}$$

then the transversality condition (4) holds.

Proof. Suppose that there exists a $\lambda^* \in (0,1)$ and a summable sequence $\{e_t\}$ such that (32) holds. Let $T \in \mathbb{Z}_+$ and $\lambda \in [\lambda^*, 1)$ and define $\left\{\hat{c}_t, \hat{M}_t^d/P_t\right\}$ by

$$\hat{c}_t = \begin{cases} c_t^* \text{ if } t < T \\ c_T^* + (1-\lambda) M_T^{d*}/P_T \text{ if } t = T \\ \lambda c_t^* \text{ if } t > T \end{cases}, \quad \hat{M}_t^d/P_t = \begin{cases} M_t^{d*}/P_t \text{ if } t < T \\ \lambda M_T^{d*}/P_T \text{ if } t = T \\ \lambda M_t^{d*}/P_t \text{ if } t > T \end{cases}. \tag{33}$$

By (2), $\left\{\hat{c}_t, \hat{M}_t^d/P_t\right\}$ is feasible. By the definition of optimality and (33)

$$\begin{aligned} &\beta^T u\left(c_T^* + (1-\lambda) M_T^{d*}/P_T, \lambda M_T^{d*}/P_T\right) - \beta^T u\left(c_T^*, M_T^{d*}/P_T\right) + \\ &\lim_{s\to\infty} \textstyle\sum_{t=T+1}^{s} \beta^t \left[u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right) - u\left(c_t^*, M_t^{d*}/P_t\right)\right] \leq 0. \end{aligned} \tag{34}$$

Therefore, by (32)

$$\lim_{s\to\infty} \begin{aligned} &\frac{\beta^T u\left(c_T^* + (1-\lambda) M_T^{d*}/P_T, \lambda M_T^{d*}/P_T\right) - \beta^T u\left(c_T^*, M_T^{d*}/P_T\right)}{1-\lambda} \leq \\ &\frac{\sum_{t=T+1}^{s} \beta^t \left[u\left(c_t^*, M_t^{d*}/P_t\right) - u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right)\right]}{1-\lambda} \leq \textstyle\sum_{t=T+1}^{\infty} e_t. \end{aligned} \tag{35}$$

Let $\lambda \to 1$. By the definition of a derivative,

$$\beta^T \left[u_c\left(c_T^*, M_T^{d*}/P_T\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right)\right] M_T^{d*}/P_T \leq \textstyle\sum_{t=T+1}^{\infty} e_t. \tag{36}$$

Letting $T \to \infty$ yields the result.

Lemma 2. Let $\rho \in \mathbb{R}$ and b_t be as in Assumption 2. Then

$$\beta^t u(\lambda c, \lambda m) \geq \lambda^\rho \left[\beta^t u(c, m) - b_t \int_\lambda^1 z^{-\rho-1} dz\right] \; \forall (c, m) \in \mathbb{R}_{++}, \forall t \in \mathbb{Z}_+. \tag{37}$$

Proof. Let $(c, m) \in \mathbb{R}_{++}$ and $t \in \mathbb{Z}_+$. Define $v(z) = \beta^t u(zc, zm)$ for $z \in (0, 1]$. By the definition of v, ρ and b_t,

$$zv'(z) = \beta^t u_c(zc, zm) zc + \beta^t u_m(zc, zm) zm \leq \rho \beta^t u(zc, zm) + b_t = \rho v(z) + b_t. \tag{38}$$

This implies

$$\begin{aligned} &\frac{d\left[z^{-\rho} v(z)\right]}{dz} = z^{-\rho} v'(z) - \rho z^{-\rho-1} v(z) \leq z^{-\rho-1} b_t \Rightarrow \\ &\int_\lambda^1 d\left[z^{-\rho} v(z)\right] \leq b_t \int_\lambda^1 z^{-\rho-1} dz \Rightarrow v(\lambda) \geq \lambda^\rho \left[v(1) - b_t \int_\lambda^1 z^{-\rho-1} dz\right]. \end{aligned} \tag{39}$$

By the definition of v, this yields the result.

To prove Proposition 2, let $\lambda^* \in (0, 1)$, $\lambda \in [\lambda^*, 1)$ and $t \in \mathbb{Z}_+$. By Lemma 2,

$$\begin{aligned} &\beta^t u\left(c_t^*, M_t^{d*}/P_t\right) - \beta^t u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right) \leq \\ &(1 - \lambda^\rho) \beta^t u\left(c_t^*, M_t^{d*}/P_t\right) + \lambda^\rho b_t \int_\lambda^1 z^{-\rho-1} dz = \\ &\beta^t u\left(c_t^*, M_t^{d*}/P_t\right) \int_\lambda^1 \rho z^{\rho-1} dz + b_t \int_\lambda^1 \lambda^\rho z^{-\rho-1} dz \leq \\ &\left|\beta^t u\left(c_t^*, M_t^{d*}/P_t\right)\right| (1 - \lambda) \max_{z \in [1, \lambda]} \rho z^{\rho-1} + |b_t| (1 - \lambda) \max_{z \in [1, \lambda]} \lambda^\rho z^{-\rho-1}. \end{aligned} \tag{40}$$

Thus,

$$\begin{aligned} \frac{\beta^t u\left(c_t^*, M_t^{d*}/P_t\right) - \beta^t u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right)}{1 - \lambda} &\leq \\ \left|\beta^t u\left(c_t^*, M_t^{d*}/P_t\right)\right| \max_{z \in [1, \lambda]} \rho z^{\rho-1} + |b_t| \max_{z \in [1, \lambda]} \lambda^\rho z^{-\rho-1} &\equiv e_t. \end{aligned} \tag{41}$$

The sequence $\{e_t\}$ is summable; hence, by Lemma 1, the proposition is proved.

Proof of Proposition 6. Let $\left\{c_t, M_t^d/P_t\right\}$ be any feasible sequence. By (16)

$$
\begin{aligned}
D &\equiv \lim_{T\to\infty}\inf \textstyle\sum_{t=0}^{T} \beta^t \left[u\left(c_t, M_t^d/P_t\right) - u\left(c_t^*, M_t^{d*}/P_t\right)\right] \\
&= \lim_{T\to\infty}\inf \textstyle\sum_{t=0}^{T} \beta^t [u\left((1+i_t)(P_{t-1}/P_t)a_{t-1}^d + y - \tau_t - a_t^d - \left(i_t P_{t-1}/P_t\right) M_{t-1}^d/P_{t-1}, M_t^d/P_t\right) \\
&\quad -u\left((1+i_t)(P_{t-1}/P_t)a_{t-1}^{d*} + y - \tau_t - a_t^{d*} - \left(i_t P_{t-1}/P_t\right) M_{t-1}^{d*}/P_{t-1}, M_t^{d*}/P_t\right)]
\end{aligned}
$$

Then by Assumption 1,

$$
\begin{aligned}
D &\leq \lim_{T\to\infty} \textstyle\sum_{t=0}^{T} \beta^t \{u_c\left(c_t^*, M_t^{d*}/P_t\right) [(1+i_t)(P_{t-1}/P_t)\left(a_{t-1}^d - a_{t-1}^{d*}\right) - \left(a_t^d - a_t^{d*}\right) \\
&\quad - \left(i_t P_{t-1}/P_t\right)\left(M_{t-1}^d/P_{t-1} - M_{t-1}^{d*}/P_{t-1}\right)] + u_m\left(c_t^*, M_t^{d*}/P_t\right)\left(M_t^d/P_t - M_t^{d*}/P_t\right)\} \\
&= \lim_{T\to\infty} \{\textstyle\sum_{t=-1}^{T-1} \beta^t u_c\left(c_{t+1}^*, M_{t+1}^{d*}/P_{t+1}\right) \beta(P_t/P_{t+1})[(1+i_{t+1})\left(a_t^d - a_t^{d*}\right) - i_{t+1}\left(M_t^d/P_t - M_t^{d*}/P_t\right)] \\
&\quad - \textstyle\sum_{t=0}^{T} \beta^t \left[u_c\left(c_t^*, M_t^{d*}/P_t\right)\left(a_t^d - a_t^{d*}\right) - u_m\left(c_t^*, M_t^{d*}/P_t\right)\left(M_t^d/P_t - M_t^{d*}/P_t\right)\right]\}.
\end{aligned}
$$

Thus by (19), (20) and the initial conditions

$$
\begin{aligned}
D &\leq -\lim_{T\to\infty} \{\beta^T \left[u_c\left(c_T^*, M_T^{d*}/P_T\right)\left(a_T^d - a_T^{d*}\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right)\left(M_T^d/P_T - M_T^{d*}/P_T\right)\right]\} \\
&= -\lim_{T\to\infty} \{\beta^T \{u_c\left(c_T^*, M_T^{d*}/P_T\right)\left(B_T^d/P_T - B_T^{d*}/P_T\right) \\
&\quad + \left[u_c\left(c_T^*, M_T^{d*}/P_T\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right)\right]\left(M_T^d/P_T - M_T^{d*}/P_T\right)\}\}.
\end{aligned}
$$

By (19), when $i_{t+1} = 0$, $u_m\left(c_T^*, M_T^{d*}/P_T\right) = 0$ and when $i_{t+1} > 0$, $u_c\left(c_T^*, M_T^{d*}/P_T\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right) > 0$; hence $[u_c\left(c_T^*, M_T^{d*}/P_T\right)$ - $u_m\left(c_T^*, M_T^{d*}/P_T\right)]M_T^d/P_T \geq 0$. By (17) and (20), $\lim_{t\to\infty} \beta^T u_c\left(c_T^*, M_T^{d*}/P_T\right) B_T^d/P_T \geq 0$. Thus

$$
D \leq \lim_{T\to\infty} \{\beta^T u_c\left(c_T^*, M_T^{d*}/P_T\right) B_T^{d*}/P_t + \left[u_c\left(c_T^*, M_T^{d*}/P_T\right) - u_m\left(c_T^*, M_T^{d*}/P_T\right)\right] M_T^{d*}/P_T\}.
$$

The right-hand-side is non-negative by (21), establishing the result.

Proof of Proposition 7. Except for Lemma 1 this follows the proof of proposition 2 in a straightforward manner. In the statement of Lemma 1, (4) is replaced by (21). The proof of Lemma 1 is now as follows.

Suppose that there exists a $\lambda^* \in (0,1)$ and a summable sequence $\{e_t\}$ such that (32) holds. Let $T \in \mathbb{Z}_+$ and $\lambda \in [\lambda^*, 1)$ and define $\left\{\hat{c}_t, \hat{M}_t^d/P_t, \hat{a}_t^d\right\}$ by

$$\hat{c}_t = \begin{cases} c_t^* \text{ if } t < T \\ c_T^* + (1-\lambda)\, a_T^{d*} \text{ if } t = T \\ \lambda c_t^* \text{ if } t > T \end{cases}, \hat{a}_t^d = \begin{cases} a_t^{d*} \text{ if } t < T \\ a_T^* \text{ if } t = T \\ \lambda a_t^* \text{ if } t > T \end{cases}, \tag{42}$$

$$\hat{M}_t^d/P_t = \begin{cases} M_t^{d*}/P_t \text{ if } t < T \\ \lambda M_T^{d*}/P_T \text{ if } t = T \\ \lambda M_t^{d*}/P_t \text{ if } t > T \end{cases}. \tag{43}$$

By (16), $\left\{\hat{c}_t, \hat{M}_t^d/P_t, \hat{a}_t^d\right\}$ is feasible. By the definition of optimality and (33)

$$\begin{aligned} &\beta^T u\left(c_T^* + (1-\lambda)\, a_T^{d*}, \lambda M_T^{d*}/P_T\right) - \beta^T u\left(c_T^*, M_T^{d*}/P_T\right) + \\ &\lim_{s\to\infty} \textstyle\sum_{t=T+1}^{s} \beta^t \left[u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right) - u\left(c_t^*, M_t^{d*}/P_t\right)\right] \leq 0. \end{aligned} \tag{44}$$

Therefore, by (32)

$$\begin{aligned} &\frac{\beta^T u\left(c_T^* + (1-\lambda)\, a_T^{d*}, \lambda M_T^{d*}/P_T\right) - \beta^T u\left(c_T^*, M_T^{d*}/P_T\right)}{1-\lambda} \leq \\ \lim_{s\to\infty} \quad &\frac{\sum_{t=T+1}^{s} \beta^t \left[u\left(c_t^*, M_t^{d*}/P_t\right) - u\left(\lambda c_t^*, \lambda M_t^{d*}/P_t\right)\right]}{1-\lambda} \leq \textstyle\sum_{t=T+1}^{\infty} e_t. \end{aligned} \tag{45}$$

Let $\lambda \to 1$. By the definition of a derivative,

$$\beta^T \left[u_c\left(c_T^*, M_T^{d*}/P_T\right) a_T^{d*} - u_m\left(c_T^*, M_T^{d*}/P_T\right) M_T^{d*}/P_T\right] \leq \textstyle\sum_{t=T+1}^{\infty} e_t. \tag{46}$$

Letting $T \to \infty$ yields that the left-hand-side is non-positive; feasibility ensures it equals zero.

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