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CONTROLLABILITY AND THE THEORY OF ECONOMIC POLICY

A further note

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We are grateful to Mr. Tondini and to another correspondent [Wenzel (1982)] for pointing out and correcting an error in section 5 of our paper [Buiter and Gersovitz (1981)] concerning necessary conditions for dynamic path controllability. Prompted by Tondini's demonstration that in the continuous time case a system is dynamically path controllable only if the number of target variables is less than or equal to the number of instrument variables, we review in this note the condition for dynamic path controllability for discrete time systems derived by Uebe (1977) and reproduced on p. 42 of our paper. We show that the rather complicated necessary and sufficient condition for dynamic path controllability given in Uebe (1977) amounts to no more than the Tinbergen static controllability condition: the state vector can track an arbitrary, given target path for two or more successive periods only if the number of instruments is no smaller than the number of state variables.

Consider the discrete time linear state equation:

$$y_\tau = Ry_{\tau-1} + Sx_\tau, \qquad \tau = 1, 2, \ldots, \tag{1}$$

where R is an $n \times n$ real constant matrix, S is an $n \times m$ real constant matrix, y an $n \times 1$ vector of state variables and x an $m \times 1$ vector of instruments. The model is called perfectly (state) controllable or dynamically path controllable for P periods starting t periods from now if after attainment of a given point,

$$y_t = y_t^*, \qquad y_t^* \quad \text{given}, \tag{2a}$$

the model can remain on a given trajectory:

$$y_{t+p}=y^*_{t+p}, \qquad y^*_{t+p} \text{ given for } p=0,1,\ldots,P-1. \tag{2b}$$

Let M be the $nP\times(t+P-1)m$ matrix given by:

$$M\equiv\begin{bmatrix} R^{t-1}S & R^{t-2}S\ldots\ldots RS & S & 0\ldots\ldots 0 \\ R^tS & R^{t-1}S\ldots\ldots R^2S & RS & S\ldots\ldots 0 \\ \vdots & \vdots \qquad \vdots & \vdots & \vdots \ddots \vdots \\ R^{t-2+P}S & R^{t-3+P}S\ldots R^PS & R^{P-1}S & R^{P-2}S\ldots S \end{bmatrix}. \tag{3}$$

Uebe shows that the system given in (1) is dynamically path controllable for P periods starting t periods from the initial date $\tau=1$ if and only if rank $(M)=nP$. While correct, this condition can be simplified greatly.

The rank of M is the same as the rank of $\tilde{M}$ defined by:

$$\tilde{M}=\begin{bmatrix} R^{t-1}S & R^{t-2}S\ldots\ldots RS & S & 0\ldots\ldots 0 \\ 0 & 0\ldots\ldots\ldots 0 & 0 & S\ldots\ldots 0 \\ \vdots & \vdots \qquad \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0\ldots\ldots\ldots 0 & 0 & 0\ldots\ldots S \end{bmatrix}.$$ [1]

Thus for $P=1$, the *path* controllability criterion reduces to the familiar dynamic *point* controllability condition that the rank of $[R^{t-1}S \quad R^{t-2}S\ldots RS \quad S]$ be equal to n. For $P\geqq 2$, the controllability criterion is that rank $(S)=n$.

This result, that for controllability or reproducibility over an interval of two periods or more ($P\geqq 2$) the Tinbergen static controllability is necessary and sufficient, can also be shown directly as follows.

Sufficiency

If rank $(S)=n$ then the system (1) is dynamically path controllable for any $t\geqq 1$, $P\geqq 1$.

[1] *Proof.* Premultiply M by the $nP\times nP$ matrix Q:

$$Q\equiv\begin{bmatrix} I & 0 & 0 & 0\ldots\ldots 0 & 0 \\ -R & I & 0 & 0\ldots\ldots 0 & 0 \\ 0 & -R & I & 0\ldots\ldots 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0\ldots & \ldots & \ldots -R & I \end{bmatrix},$$

where I is the $n\times n$ identity matrix. Clearly Q has rank nP. Since Q is a non-singular matrix and $\tilde{M}=QM$, rank $(\tilde{M})=$ rank (M).

Clearly, $y_t^* - Ry_{t-1} = Sx_t$ is solvable for x_t given any y_t^* and y_{t-1} if S has rank n. Similarly, $y_{t+p}^* - Ry_{t-1+p}^* = Sx_{t+p}$ is solvable for x_{t+p} given any y_{t+p}^* and y_{t-1+p}^*, $p = 0, 1, 2, \ldots, P-1$ if S has rank n.

Necessity

Let (1) be controllable for $P = 1$, i.e.

$$y_t = y_t^* = Ry_{t-1} + Sx_t. \tag{4}$$

For (1) to be dynamically path controllable for $P = 2$ we must have $y_{t+1} = y_{t+1}^*$, i.e.

$$y_{t+1}^* - Ry_t^* = Sx_{t+1}. \tag{5}$$

Since the n-vector y_{t+1}^* can be assigned any value, $y_{t+1}^* - Ry_t^* = Sx_{t+1}$ is solvable for x_{t+1} only if rank $(S) = n$.

Since the target values y_{t+p}^* can be assigned freely, achieving these targets exactly in successive periods simply means applying the Tinbergen criterion in each period except the first. By point controllability, the first in a sequence of successive target values (i.e. y_t^*) may be achieved even if there are fewer instruments than targets, since lagged instrument values x_{t-i}, $i = 1, 2, \ldots, t-1$, can be used to 'set up' y_t as well as the current instrument value x_t. Having achieved y_t^*, however, one can achieve $y_{t+1} = y_{t+1}^*$ for arbitrary y_{t+1}^* if and only if S has rank equal to n, the dimension of the state vector.

The exact achievement of a target value of an n-dimensional state vector for a single period may be possible with fewer instruments than targets. Reproducing a target trajectory for two or more successive periods is possible if and only if the much more restrictive Tinbergen static controllability criterion is satisfied.

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