WALRAS' LAW AND ALL THAT

Budget Constraints and Balance Sheet Constraints

in

Period Models and Continuous Time Models

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1. Introduction

This paper derives the exact financial constraints facing different economic agents in sequential temporary or momentary general equilibrium models. In doing so a number of fundamental problems in macroeconomic modelling are satisfactorily resolved for the first time. The first contribution is a new interpretation of the distinction between beginning-of-period and end-of-period models in terms of incomplete spot and forward markets for assets. In addition the paper provides the exact derivation of continuous time [or rather continuous decision making] analogues of discrete time period models. The separation of the saving decision and the portfolio allocation decision, and the appropriate definition of saving and disposable income in models with capital gains are also considered. Some unfamiliar conditions required for Walras' Law of markets to be obtainable from the summation of individual budget constraints are also derived. A small macroeconomic model is used to illustrate the various points made in the paper. Earlier work on this subject (May [1970], Chand [1973], Foley [1975], Buiter and Woglom [1977], Buiter [1975], Turnovsky [1977], Burmeister and Turnovsky [1976] [1977]) has not resolved the issues just mentioned, because it was either incomplete or contained serious errors. This is substantiated in the paper.

The major conclusions of the paper can be stated as follows: The distinction between end-of-period and beginning-of-period models lies in differing assumptions about the existence of spot markets and forward markets for assets. The beginning-of-period and the end-of-period models are not equivalent unless one assumes perfect markets and perfect foresight.

In neither model can the saving decision and the portfolio allocation be separated. Standard national income and flow of funds accounting concepts of saving and disposable income can be identified naturally in both models. By not treating income as predetermined, the following necessary condition for the aggregation of ex-ante individual (or sectoral) budget constraints to imply Walras' Law--the value of the excess demand for all goods is identically equal to zero--is brought out: current period expectations with regards to exchange ratios and non-market transfers must be homogeneous [identical] for all individuals involved. The now well-known result of May [1970], that the single budget constraint of the end-of-period discrete time model decomposes in continuous time into a balance sheet constraint for asset excess demands and a flow constraint involving transactions in all goods, is confirmed.

2. The Model

There is a uniform market period of length h for all goods, services and financial claims. The market period, or "Hicksian week" is the time interval for which prices are constant. For simplicity this market period is also assumed to have the same length as the unit periods for all other economic activities and decisions [the period of production, the period for which expectations remain constant, etc.]. At the beginning of each period, prices are determined that clear the markets for goods, services and financial claims that period. In the end-of-period model asset markets can be viewed as one-period forward markets. There exist no instantaneous spot markets for financial claims. Asset demands and supplies refer to stocks to be delivered and held at the end of the period (the beginning of the next period). In the beginning-of-period model asset markets are instantaneous spot markets. There are no one period forward markets. Asset demands and supplies refer to asset stocks to be delivered and held at the same instant. Both can be viewed as special cases of a general model in which, each period, there exists a complete set of spot and forward markets. There are three financial claims; government fiat money, one period nominally denominated government bonds, and equity: ownership claims to the residual income of corporations. There is one commodity that can be used as a private consumption good, a public consumption good or a private investment good. There is no market for existing (installed) capital goods and no capital rental market, because the only kind of capital is circulating capital. Finally there are labour services.

There are three sectors, the household sector, the business sector and the public sector. Households own all financial claims in the economy. They do not hold real reproducible capital directly in their portfolios. They consume output bought from the business sector, sell labour services to the business sector and as owners of the firms' equity receive dividends from the business sector. They receive interest on their holdings of government interest-bearing debt and pay taxes. Firms organize production. They are the custodians of the stock of real reproducible capital. They hire labour services and sell output to households, (as consumption), to the government and to themselves (as investment). Any excess of current outlays over receipts from sales is financed by issuing equity. The government purchases output produced by the business sector, raises taxes, pays interest on its debt and finances any excess of current outlays over current receipts by some conbination of borrowing and printing money.

The temporal specification of asset demand functions, price functions and price expectation functions.

In the literature on this subject, asset demand functions have been specified with one time argument (Foley [1975]) or two time arguments (Buiter and Woglom [1977] Turnovsky [1977] Burmeister and Turnovsky [1977]). I shall argue that time enters into fully specified asset demand functions [and supply functions] in four ways. The money demand function, M^d will be used to illustrate this. The complete specification of its "time structure" would be

1. $M^d(t-\Delta t_3, t, t+\Delta t_1, \Delta t_2)$ $\qquad \Delta t_1, \Delta t_2, \Delta t_3 \geq 0$.

Equation 1 describes the amount of money an economic agent plans, at time $t - \Delta t_3$, to demand at time t , for delivery at time $t + \Delta t_1$, when the length of the interval between successive markets is Δt_2 . Δt_1 will be a nonnegative integer multiple of Δt_2 . In the end-of-period model, $\Delta t_1 = \Delta t_2 = h$, and it is only necessary to consider the case where $\Delta t_3 = 0$. I shall therefore only deal with asset demands in the form given by 2).

2. $M^d(t,t,t+h,h)$

In the beginning-of-period model, $\Delta t_1 = 0$ and $\Delta t_2 = h$. It is only necessary to consider the cases where $\Delta t_3 = 0$ and where $\Delta t_3 = h$. I shall therefore deal only with asset demands in the form 3a) and 3b).

3a. $M^d(t,t,t,h)$

3b. $M^d(t,t+h,t+h,h)$

The first time argument, $t-\Delta t_3$, needs to be considered explicitly if one is to make sense of the beginning-of-period specification, and is important when I study the equivalence of the limiting forms [as $h \to 0$] of the end-of-period and the beginning-of-period model.

Δt_2 needs to be considered explicitly because a basic feature of the model is altered when markets open more frequently (Hellwig [1975]).

Actual asset prices can be fully characterized as regards their time structure in the following way

4. $p_i(t, t+\Delta t_1, \Delta t_2)$

Equation 4) gives the price at time t for asset i to be delivered at time $t+\Delta t_1$ when the length of the interval between successive markets is Δt_2. In the end-of-period model I shall only deal with:

5. $p_i(t, t+h, h)$

In the beginning-of-period model I only consider:

6. $p_i(t, t, h)$

Asset price expectations have the same "time structure" as the asset demand functions. Expectations are single-valued. The general specification is given in equation 7

7. $p_i^*(t-\Delta t_3, t, t+\Delta t_1, \Delta t_2)$ [1]

This is the price expected, at time $t-\Delta t_3$ to prevail at time t for a unit of asset i to be delivered at time $t+\Delta t_1$ when the length of the interval between market periods is Δt_2.

In the end-of-period model two price expectations are relevant. They are

8a. $p_i^*(t, t, t+h, h)$,

the price expected at time t to be established at time t for a unit of asset i to be delivered at time $t+h$ when the interval between successive markets has length h.

8b. $p_i^*(t, t+h, t+2h, h)$,

the price expected, at the beginning of period t to prevail during the next period for a unit of asset i to be delivered at the end of the next period when the interval between successive markets has length h.

In the beginning-of-period model the two relevant price expectations are given in 9a) and 9b)

9a. $p_i^*(t,t,t,h)$

and

$p_i^*(t,t+h,t+h,h)$

The interpretation of these is straightforward. I shall assume that expectations about current period prices are correct. We therefore have, in the end-of-period model:

10a. $p_i^*(t,t,t+h,h) = p_i(t,t+h,h)$

and in the beginning-of-period model:

10b. $p_i^*(t,t,t,h) = p_i(t,t,h)$

It is essential to make the specification of the time structure of our functions as detailed as in equations 1), 4) and 7). Equation 5) and 6) remind one that the current period price of an asset in the end-of-period model is quite different from the current period price of the asset in the beginning-of-period model. The former is the price of a forward contract, the latter of a spot contract. Equations 3b), 8b) and 9b) remind one that h can enter end-of-period demand functions and expectation functions in 3 ways: as a "forecast interval" ($h=\Delta t_3$), as the maturity of the forward contract ($h=\Delta t_1$), and as the length of the interval between successive markets ($h=\Delta t_2$). In the beginning-of-period model the maturity of the forward contract is identically zero, but $\Delta t_2=h$ here as well.

Notation:

M^d nominal demand for money

M^s nominal supply of money

$M(t)$ nominal stock of money at the beginning of period t

B^d nominal demand for bonds

B^s nominal supply of bonds

$B(t)$ nominal stock of bonds at the beginning of period t

E^d number of shares of equity demanded

E^s number of shares of equity supplied

$E(t)$ number of shares of equity in existence at the beginning of period t

$N^d(h,t)$ demand for labour during period t

$N^s(h,t)$ supply of labour during period t

$N(h,t)$ amount of labour employed during period t

$C^d(h,t)$ consumption demand during period t

$C(h,t)$ actual consumption during period t

$Q^s(h,t)$ supply of output from new production during period t

$Q(h,t)$ actual production during period t

$G^d(h,t)$ government consumption demand during period t

$G(h,t)$ government consumption during period t

$\Pi^H(h,t)$ nominal value of dividends expected by households during period t

$\Pi^f(h,t)$ nominal value of dividends firms plan to pay out during period t

$\Pi(h,t)$ nominal value of dividends paid out during period t

$T^H(h,t)$ nominal value of taxes households plan to pay during period t

$T^g(h,t)$ nominal value of taxes the government plans to raise during period t

$T(h,t)$ nominal value of taxes raised during period t

$I^d(h,t)$ investment demand during period t

$I(h,t)$ investment during period t

p_E money price of equity

p_E^* expected money price of equity

$r(t,t,h)$ instantaneous nominal interest rate established at t, to be paid on bonds during period t

$w(t,t,h)$ nominal wage during period t [money price established at t of labour services to be delivered between t and $t+h$].

All asset demand functions and the price and expected price of equity have the time arguments specified in the previous section. I shall not be concerned with the special problems associated with capital accumulation in a one-commodity world, as this would add nothing to our understanding of the issues analyzed in this paper. I shall therefore assume that all capital is circulating capital with a period of circulation equal to h. With output non-durable, the only relevant price of output is the spot price.

$p(t,t,h)$ money price established at t of output to be delivered between t and $t+h$.

p^* expected price of output

All functions are assumed to be differentiable when needed.

3. An End-of-Period Model

The discrete time model

The ex-ante budget constraint is the consistency requirement equating the value of all planned uses of funds with the value of all expected sources of funds. The ex-ante household budget constraint can be written as

11. $$M(t)+B(t)+p_E(t,t+h,h)E(t)+w(t,t,h)N^S(h,t)+\Pi^H(h,t)+r(t,t,h)B(t)h \equiv M^d(t,t,t+h,h)+B^d(t,t,t+h,h)+p_E(t,t+h,h)E^d(t,t,t+h,h)+p(t,t,h)C^d(h,t)+T^H(h,t).$$

Instead of assuming the income flow to be fixed and known, as in Turnovsky [1977], it is broken down into expected income from employment

wN^s and expected dividend receipts. Clearly, the budget constraint should include the net planned purchases or sales of financial claims during period t [$M^d(t,t,t+h,h)-M(t)$, $B^d(t,t,t+h,h)-B(t)$ and $E^d(t,t,t+h,h)-E(t)$] valued at the prices that prevail during period t, i.e. in the case of equity: $p_E(t,t+h,h)$. Turnovsky's analysis [1977] is flawed from the start because he evaluates planned equity sales or purchases not at the price at which the transactions are made, but at the price expected at t, to prevail at $t+h$, $p_E^*(t,t+h,t+2h,h)$.

Rationality requires that asset demands be real demands, i.e. demands for a certain amount of purchasing power over consumable commodities. As the end-of-period asset demands refer to asset stocks to be held at the end of the period, the relevant prices for translating real asset demands into nominal asset demands are next period's expected prices. Let $\ell(t,t,t+h,h)$ be the real end-of-period demand for money balances, $b(t,t,t+h,h)$ the real end-of-period demand for bonds and $j(t,t,t+h,h)$ the real end-of-period demand for equity. Then:

12a. $M^d(t,t,t+h,h) = p^*(t,t+h,t+h,h)\ell(t,t,t+h,h)$

12b. $B^d(t,t,t+h,h) = p^*(t,t+h,t+h,h)b(t,t,t+h,h)$

12c. $E^d(t,t,t+h,h) = \dfrac{p^*(t,t+h,t+h,h)}{p_E^*(t,t+h,t+2h,h)}\; j(t,t,t+h,h)$

The ex-post household budget constraint or flow of funds identity is obtained by replacing all ex-ante [planned or expected] variables in 11) by their realized values.

11'. $M(t)+B(t)+p_E(t,t+h,h)E(t)+w(t,t,h)N(h,t)+\Pi(h,t)+r(t,t,h)B(t)h$

$\equiv M(t+h)+B(t+h)+p_E(t,t+h,h)E(t+h)+p(t,t,h)C(h,t)-T(h,t)$.

The ex-ante budget constraint of the business sector is:

13. $p(t,t,h)Q^s(h,t)+p_E(t,t+h,h)[E^s(t,t,t+h,h)-E(t)] \equiv w(t,t,h)N^d(h,t)$
$+ \Pi^f(h,t)+p(t,t,h)I^d(h,t).$

The firms' sources of funds are the proceeds from the sale of its output and of its equity. Its uses of funds are purchases of labour services, dividend payments and investment in circulating capital. The ex-post flow of funds identity of the business sector is:

$$p(t,t,h)Q(h,t)+p_E(t,t+h,h)[E(t+h)-E(t)] \equiv w(t,t,h)N(h,t)+\Pi(h,t)+p(t,t,h)I(h,t)$$

The public sector ex-ante budget constraint is:

14. $M^s(t,t,t+h,h)-M(t)+B^s(t,t,t+h,h)-B(t)+T^g(h,t)\equiv p(t,t,h)G^d(h,t)+r(t,t,h)B(t)h$

Ex-post the public sector flow of funds identity is:

15. $M(t+h)-M(t)+B(t+h)-B(t)+T(h,t)\equiv p(t,t,h)G(h,t)+r(t)B(t)h$

Summing the ex-ante budget constraints of the three sectors and rearranging terms we obtain

16. $M^s(t,t,t+h,h)-M^d(t,t,t+h,h)+B^s(t,t,t+h,h)-B^d(t,t,t+h,h)$
$+p_E(t,t+h,h)[E^s(t,t,t+h,h)-E^d(t,t,t+h,h)]+w(t,t,h)[N^s(h,t)-N^d(h,t)]$
$+p(t,t,h)[Q^s(h,t)-C^d(h,t)-I^d(h,t)-G^d(h,t)]\equiv\Pi^f(h,t)-\Pi^H(h,t)+T^H(h,t)-T^g(h,t)$

The market equilibrium conditions for this five good economy are:

17a. $M^s(t,t,t+h,h)=M^d(t,t,t+h,h)$ $[=M(t+h)]$

17b. $B^s(t,t,t+h,h)=B^d(t,t,t+h,h)$ $[=B(t+h)]$

17c. $E^s(t,t,t+h,h)=E^d(t,t,t+h,h)$ $[=E(t+h)]$

17d. $N^s(h,t)=N^d(h,t)$ $[=N(h,t)]$

17e. $Q^s(h,t)=C^d(h,t)+I^d(h,t)+G^d(h,t)$ $[Q^s=Q, C^d=C, I^d=I, G^d=G]$

The sum of the values of the excess demands for all goods, X(h,t) is

18. $X(h,t) \equiv M^s(t,t,t+h,h) - M^d(t,t,t+h,h) + B^s(t,t,t+h,h) - B^d(t,t,t+h,h)$
$+ p_E(t,t+h,h)[E^s(t,t,t+h,h) - E^d(t,t,t+h,h)] + w(t,t,h)[N^s(h,t) - N^d(h,t)]$
$+ p(t,t,h)[Q^s(h,t) - C^d(h,t) - I^d(h,t) - G^d(h,t)]$

Walras' Law of Markets is the proposition that $X(h,t) \equiv 0$. From 16) and 18) we see that this is only implied by the sectoral budget constraints if

$\Pi^f - \Pi^H + T^H - T^g \equiv 0$

Offsetting values for $\Pi^f - \Pi^H$ and $T^H - T^g$ could bring this about, but the most general and attractive condition for 16) to imply $X \equiv 0$ is $\Pi^f - \Pi^H = T^H - T^g = 0$. The dividends households expect to receive from firms equal the dividends firms plan to pay out. The taxes households expect to pay to the government equal the taxes the government plans to levy. Thus homogeneous expectations with respect to variables entering the budget constraints of several economic agents during the current period will in general be required to ensure that Walras' Law holds. By treating income (and output) as predetermined, this aspect of the relationship between ex-ante individual budget constraints and Walras' Law has been suppressed in the rest of the literature [e.g. Turnovsky (1977)]. In conventional Arrow-Debreu general equilibrium analysis, taxes and transfers are ignored and the same profit term is always entered into both household and firm budget constraints. Note that, for every market transaction, the price at which the buyer plans to purchase and the price at which the seller plans to sell should also be the same [after allowing for transactions costs and indirect taxes or subsidies]. If this were not the case, excess demands and supplies for the same good could not be summed using a single price and Walras' Law would have no meaning.

Dividends and taxes in our simple model extend to non-market transactions this requirement of homogeneity of expectations as regards current period prices. In its absence, the fact that individual agents satisfy their ex-ante budget constraints at their perceived prices does not imply Walras' Law of Markets, which sums all individual excess demands valued at a common price vector. In this paper and in the rest of the literature homogeneity of both current and future price expectations is assumed.

<u>The continuous trading version of the end-of-period model</u>

The primitive flow variables of the model are $N^d, N^s, N, C^d, C, Q^s,$ $Q, G^d, G, \Pi^H, \Pi^f, \Pi, T^H, T^g, T, I^d$ and I.

For any flow variable $Z(h,t)$ we define:

$$Z(h,t) \equiv \int_t^{t+h} z(\tau,h,t)d\tau$$

From the fundamental theorem of the calculus we have:

$$\lim_{h\to 0} \frac{Z(h,t)}{h} = z(t,0,t) = z(t)$$

It is also clear that

$$\lim_{h\to 0} Z(h,t)=0$$

When we let h go to zero, we obtain from the household budget constraint (11):

19. $M(t)+B(t)+p_E(t,t,0)E(t) \equiv M^d(t,t,t,0)+B^d(t,t,t,0)+p_E(t,t,0)E^d(t,t,t,0)$

When the length of the delivery period is zero and successive market periods are arbitrarily close together, the value of all asset stocks we plan to carry over into the next period is constrained to be equal to the value of the endowment of all assets at the beginning of the period. 19) is the household sector balance sheet constraint. Combining 11) and 19), rearranging terms and dividing by h yields:

$$\frac{[p_E(t,t+h,h)-p_E(t,t,0)]E(t)]}{h} + \frac{w(t,t,h)N^s(h,t)}{h} + \frac{\Pi^H(h,t)}{h} + \frac{r(t,t,h)B(t)h}{h}$$

$$\equiv \frac{M^d(t,t,t+h,h)-M^d(t,t,t,0)}{h} + \frac{B^d(t,t,t+h,h)-B^d(t,t,t,0)}{h}$$

$$+ p_E(t,t+h,h)\frac{[E^d(t,t,t+h,h)-E^d(t,t,t,0)]}{h} + \frac{[p_E(t,t+h,h)-p_E(t,t,0)]E^d(t,t,t,0)}{h}$$

$$+ \frac{p(t,t,h)C^d(h,t)}{h} + \frac{T^H(h,t)}{h}$$

Taking the limit of this as $h \to 0$ yields:

20. $$M_3^d(t,t,t,0)+M_4^d(t,t,t,0)+B_3^d(t,t,t,0)+B_4^d(t,t,t,0)+p_E(t,t,0)[E_3^d(t,t,t,0)+E_4^d(t,t,t,0)]$$
$$+[p_{E,2}(t,t,0)+p_{E,3}(t,t,0)][E^d(t,t,t,0)-E(t)]\equiv w(t,t,0)n^s(t)+\pi^H(t)+r(t,t,0)B(t)-\tau^H(t)$$
$$-p(t,t,0)c^d(t)$$

When h changes one of the effects is that the length of the delivery period or the maturity of the forward contract changes. This is caught by the third partial derivatives of the asset demand functions. M_3^d, B_3^d and E_3^d . Keeping t , the time at which the market takes places constant, and keeping the frequency with which markets open [the fourth argument of the asset demand functions] constant, how does the demand for assets to be delivered and held at some future point in time vary as that future point in time varies? The fourth partial derivatives of the asset demand functions show how, keeping everything else constant, asset demands expressed at time t for assets to be held at some fixed future date, will vary when markets open more or less frequently. The first and second partial derivatives of the asset demand functions will not assume importance until we consider the beginning-of-period model, but can be interpreted here briefly. Take e.g. the demand for money. $M_1^d(t-\Delta t_3,t,t+\Delta t_1,\Delta t_2)$ measures one dimension of the revision of asset demand plans. It gives the rate at which the amount of money changes that an economic agent plans, at time $t-\Delta t_3$ to demand at time t, for delivery at time $t-\Delta t_1$ when the market period has length Δt_2, as this planning date $t-\Delta t_3$ changes. $M_2^d(t-\Delta t_3,t,t+\Delta t_1,\Delta t_2)$ measures a second dimension of the revision of asset demand plans. The planning date, the delivery date and the market interval are kept constant, and M_2^d therefore measures the rate at which the demand for money changes when the market date varies. The best measure of the rate at which the demand for money to be delivered at some given future date is revised when we get closer to that date is $M_1^d+M_2^d$. Note also that $\frac{d}{dt}M^d=M_1^d+M_2^d+M_3^d$. Earlier studies have derived continuous time budget constraints from discrete end-of-period models that involved terms equivalent to our M_3^d, B_3^d and E_3^d. The fourth partial derivatives of the asset demand functions have been uniformly ignored. Similarly, nobody has included a term like $[p_{E,2}(t,t,0)+p_{E,3}(t,t,0)][E^d(t,t,0)-E(t)]$. The second partial derivative of the price functions, e.g. $p_{E,2}$ measure the rate at which, at a given market date and with a given market period, the price of a unit of some good to be delivered at some future date varies with that future delivery date. The third partial, e.g. $p_{E,3}$ measures the rate at which the price established at a given point in time for a forward delivery of a given maturity varies when the frequency with which markets open varies. The first partial derivative measures the rate at which the price for forward delivery at a given future date varies as length of the period between the market and that future date varies. The term $(p_{E,2}+P_{E,3})(E^d-E)$ will vanish in a continuous market clearing model when households are always in portfolio equilibrium. When that is not the case, and there are net instantaneous planned sales or purchases of equity one has to allow in the budget constraint for the fact that the price of equity may vary when h varies. It should be noted that the single household budget constraint in the discrete time case [equation 11] generates both a balance sheet constraint [equation 19] and a flow budget constraint in the continuous time case.

The ex-post flow of funds identity for the household sector in continuous time can easily be seen to be:

$$\dot{M}(t)+\dot{B}(t)+p_E(t,t,0)\dot{E}(t)\equiv w(t,t,0)n(t)+\pi(t)+r(t,t,0)B(t)-\tau(t)-p(t,t,0)c(t)$$

Proceeding in the same manner used for the household sector we obtain for the business sector a balance sheet constraint and a flow of funds constraint [using 13)].

21. $E^s(t,t,t,0)=E(t)$

22. $-p_E(t,t,0)[E_3^s(t,t,t,0)+E_4^s(t,t,t,0)]\equiv -w(t,t,0)n^d(t)-\pi^f(t)-p(t,t,0)i^d(t)$
$+p(t,t,0)q^s(t)$

The ex-post flow of funds identity for the corporate sector is:

$$-p_E(t,t,0)\dot{E}(t)\equiv -w(t,t,0)n(t)-\pi(t)-p(t,t,0)i(t)+p(t,t,0)q(t)$$

For the public sector we obtain from equation 14 a continuous time balance sheet constraint and budget constraint

23. $M^s(t,t,t,0)+B^s(t,t,t,0)\equiv M(t)+B(t)$

24. $p(t,t,0)g^d(t)+r(t,t,0)B(t)\equiv\tau^g(t)+M_3^s(t,t,t,0)+M_4^s(t,t,t,0)+B_3^s(t,t,t,0)+B_4^s(t,t,t,0)$

The ex-post flow of funds identity for the public sector is

$$g(t)+r(t,t,0)B(t)\equiv\tau(t)+\dot{M}(t)+\dot{B}(t)$$

Aggregating over the three sectors we obtain the economy-wide balance sheet constraint and flow budget constraint.

25. $$M^s(t,t,t,0)-M^d(t,t,t,0)+B^s(t,t,t,0)-B^d(t,t,t,0)$$
$$+p_E(t,t,t,0)[E^s(t,t,t,0)-E^d(t,t,t,0)]\equiv 0$$

26. $$M_3^d(t,t,t,0)+M_4^d(t,t,t,0)-M_3^s(t,t,t,0)-M_4^s(t,t,t,0)+B_3^d(t,t,t,0)+B_4^d(t,t,t,0)$$
$$-B_3^s(t,t,t,0)-B_4^s(t,t,t,0)+p_E(t,t,0)[E_3^d(t,t,t,0)+E_4^d(t,t,t,0)-E_3^s(t,t,t,0)-E_4^s(t,t,t,0)]$$
$$+[p_{E,2}(t,t,0)+p_{E,3}(t,t,0)][E^d(t,t,t,0)-E(t)]+w(t,t,0)(n^d(t)-n^s(t)$$
$$+p(t,t,0)\left(c^d(t)+g^d(t)+i^d(t)-q^s(t)\right)\equiv\tau^H(t)-\pi^f(t)+\tau^g(t)-\tau^H(t)$$

The similarity to the discrete budget constraint [equation 16] is clear. With homogeneous expectations, the r.h.s.of equation 26 is zero and the continuous time analogue of Walras' Law holds.

The continuous time market clearing equations are:

27a. $M^s(t,t,t,0)=M^d(t,t,t,0)$ $\quad$ $[=M(t)]$

27b. $B^s(t,t,t,0)+B^d(t,t,t,0)$ $\quad$ $[=B(t)]$

27c. $E^s(t,t,t,0)=E^d(t,t,t,0)$ $\quad$ $[=E(t)]$

27d. $n^s(t)=n^d(t)$ $\quad$ $[=n(t)]$

27e. $q^s(t)=c^d(t)+i^d(t)+g^d(t)$ $\quad$ $[q^s=q, c^d=c, i^d=i, g^d=g]$

Saving, wealth and disposable income

All of the preceding analysis was conducted without making use of the concepts of saving, actual and target wealth (or net worth) and disposable income. The entire argument was expressed in terms of "primitive" concepts like demand and supply functions and endowments. "Derived" or Secondary" concepts like saving, net worth and disposable income are useful only to the extent that they permit the information contained in the balance sheet and flow of funds constraints to be presented in a more familiar, economical or otherwise attractive manner. They do not add in any substantive way to our understanding.

A clear implication of the end-of-period asset demand model is that the portfolio allocation decision cannot be treated separately from the saving decision. The decision on how much additional wealth to carry over to the next period and the decision on how to allocate this planned increase in wealth over the entire menu of assets in the portfolio are inextricably intertwined. This is true both in the discrete period model [equation 11] and in the continuous time case [equations 19 and 20]. In the continuous time case, once we specify the instantaneous stock asset demand functions [$M^d(t,t,t,0)$ etc.] we have at the same time specified the flow asset demands [$M_3^d(t,t,t,0)+M_4^d(t,t,t,0)$ etc.] i.e. the flow of saving. This is demonstrated in somewhat greater detail below.

Let $Y(h,t)$ denote expected nominal disposable income during period t and $S(h,t)$ the nominal value of planned personal saving during period t. The national income accounting conventions state that

28. $Y(h,t) \equiv p(t,t,h)C^d(h,t)+S(h,t)$

With $\bar{Y}(h,t)$ denoting actual nominal disposable income during period t and $\bar{S}(h,t)$ realized personal saving during period t we also have

29. $\bar{Y}(h,t) \equiv p(t,t,h)C(h,t)+\bar{S}(h,t)$

On our model there is no durable commodity. Again following common national income and flow-of-funds accounting practice we define nominal planned personal saving to be the money value of net planned purchases of financial claims.

The real value of these savings is therefore not in general the same as the planned increase in real wealth. The latter is the sum of real planned saving and expected capital gains or losses on assets to be carried over to the next period.

30. $S(h,t)\equiv M^{d}(t,t,t+h,h)-M(t)+B^{d}(t,t,t+h,h)-B(t)+p_{E}(t,t+h,h)[E^{d}(t,t,t+h,h)-E(t)]$

Equivalently, substituting from 12a) b) and c) we have:

31. $S(h,t)\equiv p^{*}(t,t+h,t+h,h)\ell(t,t,t+h,h)-M(t)+p^{*}(t,t+h,t+h,h)b(t,t,t+h,h)-B(t)$

$$+p_{E}(t,t+h,h)\left[\frac{p^{*}(t,t+h,t+h,h)}{p_{E}^{*}(t,t+h,t+2h,h)}\,j\,(t,t,t+h,h)-E(t)\right]$$

From 28), 34) and 11) it then follows that

32a. $Y(h,t)\equiv w(t,t,h)N^{s}(h,t)+\Pi^{H}(h,t)+r(t,t,h)B(t)h-T^{H}(h,t)$

Actual disposable income can then easily be seen to be:

32b. $\bar{Y}(h,t)\equiv w(t,t,h)N(h,t)+\Pi(h,t)+r(t,t,h)B(t)h-T(h,t)$

Realized saving is:

$\bar{S}(h,t)\equiv M(t+h)-M(t)+B(t+h)-B(t)+p_{E}(t,t+h,h)[E(t+h)-E(t)]$

32a) and 32b) represent the standard definition of personal disposable income in a closed economy. This is merely an accounting exercise. Whether this definition of income or, as argued by Turnovsky [1977], a modified income concept which allows for capital gains and losses is the appropriate argument to be included in the consumption function or in any other behavioural function is a matter of substantive economic theory quite distinct from the consistency arguments considered here. Like most economists I would wish to include neither 32) nor Turnovsky's capital-gains-adjusted income concept in the consumption function but rather some form of permanent or life-time income. This bears no simple relationship to any notion of current disposable income. Good bookkeeping is a prerequisite for substantive theory but not a substitute for it. Let $A(t,t+h,h)$ denote the real wealth households, at time t, plan to hold at time $t+h$, when the length of the interval between market periods is h. $A(t)$ denotes real household wealth at the beginning of period t.

33. $$A(t) \equiv \frac{M(t)}{p(t,t,h)} + \frac{B(t)}{p(t,t,h)} + \frac{p_E(t,t+h,h)E(t)}{p(t,t,h)}$$

34. $$A^d(t,t+h,h) \equiv \ell(t,t,t+h,h) + b(t,t,t+h,h) + j(t,t,t+h,h)$$

By substituting into 34) from 2a), b) and c) and using 30) we obtain

35a. $$A^d(t,t+h,h) - A(t) \equiv \frac{S(h,t)}{p(t,t,h)} - \left[\frac{p^*(t,t+h,t+h,h) - p(t,t,h)}{p(t,t,h)}\right]\left[\frac{M^d(t,t,t+h,h) + B^d(t,t,t+h,h)}{p^*(t,t+h,t+h,h)}\right]$$
$$+ \left[\frac{p_E^*(t,t+h,t+2h,h) - p_E(t,t+h,h)}{p_E(t,t+h,h)} - \frac{p^*(t,t+h,t+h,h) - p(t,t,h)}{p(t,t,h)}\right]\frac{E^d(t,t,t+h,h)\, p_E(t,t+h,h)}{p^*(t,t+h,t+h,h)}$$

or

35b. $$A^d(t,t+h,h) - A(t) \equiv \frac{S(h,t)}{p(t,t,h)} - \left[\frac{p^*(t,t+h,t+h,h) - p(t,t,h)}{p(t,t,h)}\right][\ell(t,t,t+h,h) + b(t,t,t+h,h)]$$
$$+ \left[\frac{p_E^*(t,t+h,t+2h,h) - p_E(t,t+h,h)}{p_E(t,t+h,h)} - \frac{p^*(t,t+h,t+h,h) - p(t,t,h)}{p(t,t,h)}\right] j(t,t,t+h,h)\frac{p_E(t,t+h,h)}{p_E^*(t,t+h,t+2h,h)}$$

The planned change in real private net worth over period t is the real value of net planned purchases of financial claims plus the capital gains or minus the capital losses expected to be incurred on the assets that are to be carried over into the next period. These asset stocks are not the beginning-of-period stocks $M(t)$, $B(t)$ and

and E(t), but the asset stocks households plan, at t, to hold at t+h : $M^d(t,t,t+h,h)$ etc. Ex-post the actual change in real wealth is given by:

$$A(t+h)-A(t) \equiv \frac{M(t+h)}{p(t+h,t+h,h)} + \frac{B(t+h)}{p(t+h,t+h,h)} + E(t+h)\frac{p_E(t+h,t+2h,h)}{p(t+h,t+h,h)} - \frac{M(t)}{p(t,t,h)}$$

$$- \frac{B(t)}{p(t,t,h)} - \frac{E(t)p_E(t,t+h,h)}{p(t,t,h)} \equiv \frac{\bar{S}(h,t)}{p(t+h,t+h,h)} - \left[\frac{p(t+h,t+h,h)-p(t,t,h)}{p(t,t,h)}\right]\left[\frac{M(t+h)+B(t+h)}{p(t+h,t+h,h)}\right]$$

$$+\left[\frac{p_E(t+h,t+2h,h)-p_E(t,t+h,h)}{p_E(t,t+h,h)} - \frac{[p(t+h,t+h,h)-p(t,t,h)]}{p(t,t,h)}\right]\frac{E(t+h)p_E(t,t+h,h)}{p(t+h,t+h,h)}$$

We can obtain the continuous trading version of equations 28)-35) through the same kind of limiting process used before. We now also require the assumption that the expectation of a current period price is the actual price, i.e. $p^*(t,t,t,h)=p(t,t,h)$ and $p_E^*(t,t,t+h,h)=p_E(t,t+h,h)$. In the limit as $h \to 0$ we get:

36. $y(t) \equiv p(t,t,0)c^d(t)+s(t)$

37. $\bar{y}(t) \equiv p(t,t,0)c(t)+\bar{s}(t)$

38. $s(t) \equiv M_3^d(t,t,t,0)+M_4^d(t,t,t,0)+B_3^d(t,t,t,0)+B_4^d(t,t,t,0)+p_E(t,t,0)[E_3^d(t,t,t,0)-E_4^d(t,t,t,0)]$

$+ [p_{E,2}(t,t,0)+p_{E,3}(t,t,0)][E^d(t,t,t,0)-E(t)]$

An equivalent representation of 38) in terms of the real asset demand functions is given in 39)

39. $s(t) \equiv p(t,t,0)\big(\ell_3(t,t,t,0)+\ell_4(t,t,t,0)+b_3(t,t,t,0)+b_4(t,t,t,0)+j_3(t,t,t,0)+j_4(t,t,t,0)\big)$

$+ [p_2^*(t,t,t,0)+p_3^*(t,t,t,0)+p_4^*(t,t,t,0)][\ell(t,t,t,0)+b(t,t,t,0)+j(t,t,t,0)]$

$- [p_{E,2}^*+2p_{E,3}^*+p_{E,4}^*]\ \frac{p(t,t,0)}{p_E(t,t,0)}\ j(t,t,t,0)$

$+ [p_{E,2}(t,t,0)+p_{E,3}(t,t,0)][\frac{p(t,t,0)}{p_E(t,t,0)}\ j(t,t,t,0)-E(t)]$ 2

Equation 39 translates the saving plan expressed in terms of planned rates of change of physical asset stocks into planned rates of change in real asset stocks. Expected capital gains or losses on planned holdings of assets have to be allowed for. [Note that next period's expected price of equity is the expected price for a delivery 2 periods from now]. As h changes the price for delivery h time units from now can be expected to change. The $(p_{E,2}+p_{E,3})(E^d-E)$ term allows for this in 38) and 39).

From the balance sheet constraint we note that as $h \to 0$, $A^d(t,t,0)=A(t)$. The continuous time analogue of equation 35a therefore becomes.

$$A_2^d(t,t,0)+A_3^d(t,t,0) \equiv \frac{s(t)}{p(t,t,0)} - \frac{[p_2^*(t,t,t,0)+p_3^*(t,t,t,0)]}{p(t,t,0)} \frac{[M^d(t,t,t,0)+B^d(t,t,t,0)]}{p(t,t,0)}$$

$$+ \left[\frac{p_{E,2}^*(t,t,t,0)+p_{E,3}^*(t,t,t,0)}{p_E(t,t,0)} - \frac{(p_2^*(t,t,t,0)+p_3^*(t,t,t,0))}{p(t,t,0)}\right] \frac{E^d(t,t,t,0)p_E(t,t,0)}{p(t,t,0)}$$

The rate at which households, at time t , plan to increase their real wealth is the sum of the real value of the rate of saving at t plus any expected capital gains or minus any expected capital losses due to changes in the price of equity and in the general price level. These capital gains or losses are incurred on the asset holdings households are planning to carry over to the "next" instantaneous market period. Equations 30, 31, 38 and 39 make it clear that the portfolio allocation and wealth accumulation decisions are not independent and cannot be separated in either the discrete time version or the continuous time version of the end-of-period model. In the next section I shall show that, contrary to widely held views, the same applies to a properly specified beginning-of-period model.

4. A Beginning-of-Period Model

In the discrete time beginning-of-period model, contracts are made at time t for instantaneous delivery of asset stocks. This has been interpreted as implying that decisions on how much to add to one's wealth during period t-t+h, are taken without regard for the asset composition of this net increase or decrease in wealth. This way of dichotomizing the portfolio allocation decision and the saving decision has strong proponents, e.g. Foley [1975] and strong detractors, e.g. Tobin [1977]. The latter writes: "Some authors consider a beginning-of-period stock equilibrium in which existing asset supplies are priced, followed by a within-period "flow equilibrium" in which asset accumulations, among other variables, are determined. This tortured construction makes no sense to me."[3] I shall argue that in the beginning-of-period model too, the household sector [and the other sectors as well] have to satisfy an ex-ante consistency requirement relating planned changes in asset holdings, income flows and consumption. Unlike the end-of-period model, however, the planned future holding of an asset is not represented by current period market demand for future delivery, but by planned future spot asset demand. As the only asset markets are instantaneous spot markets, planned future asset holdings cannot be made effective in markets during the current period. Rather than facing a market budget constraint, as in the end-of-period model, household current and planned future spot transactions have to satisfy an internal or "psychological" consistency requirement.

The first consistency requirement for the household sector in the beginning-of-period model is that the balance sheet constraint be satisfied. This reflects the basic assumption of beginning-of-period models that current period flows of income and expenditure cannot add to or diminish the asset stocks that are to be priced and allocated in the instantaneous asset markets.[4]

40. $M(t)+B(t)+p_E(t,t,h)E(t)\equiv M^d(t,t,t,h)+B^d(t,t,t;h)+p_E(t,t,h)E^d(t,t,t,h)$

The "psychological" budget constraint is:

41. $M^d(t,t,t,h)+B^d(t,t,t,h)+p_E^*(t,t+h,t+h,h)E^d(t,t,t,h)+w(t,t,h)N^s(h,t)$
$+\Pi^H(h,t)+r(t,t,h)B^d(t,t,t,h)h \equiv M^d(t,t+h,t+h,h)+$
$B^d(t,t+h,t+h,h)+p_E^*(t,t+h,t+h,h)E^d(t,t+h,t+h,h)$
$+p(t,t,h)C^d(h,t)+T^H(h,t)$

The expected dollar value at the beginning of period t+h, (valued at the spot prices expected to rule at t+h) of the desired beginning-of-period asset stocks in period t, plus expected money income during period t, minus taxes and minus planned consumption during period t should be equal to the value of the assets individuals plan, at t, to demand at time t+h. Note that $M^d(t,t+h,t+h,h)$ etc. are not market demands. There is no market in period t for assets to be delivered in period t+h. $M^d(t,t+h,t+h,h)$ represents the plans at time t for purchases of money balances in the spot market at t+h. Only with perfect foresight will we have $M^d(t,t+h,t+h,h) \equiv M^d(t+h,t+h,t+h,h)$. The relevant real asset demand functions are:

$$M^d(t,t+h,t+h,h) \equiv p^*(t,t+h,t+h,h)\ell(t,t+h,t+h,h)$$

$$B^d(t,t+h,t+h,h) \equiv p^*(t,t+h,t+h,h)b(t,t+h,t+h,h)$$

$$E^d(t,t+h,t+h,h) \equiv \frac{p^*(t,t+h,t+h,h)}{p_E^*(t,t+2h,t+2h,h)}\, j(t,t+h,t+h,h)$$

The ex-post budget constraint for the household sector is:

41'. $M(t)+B(t)+p_E(t+h,t+h,h)E(t)+w(t,t,h)N(h,t)+\Pi(h,t)+r(t,t,h)B(t)h$
$\equiv M(t+h)+B(t+h)+p_E(t+h,t+h,h)E(t+h)+p(t,t,h)C(h,t)+T(h,t)$

Note that this is only the same as the ex-post budget constraint of the end-of-period 11') model if $p_E(t+h,t+h,h)=p_E(t,t+h,h)$. Today's one-period forward price of equity will equal the spot price of equity next period if markets are efficient and there is perfect foresight. This becomes obvious when we envisage an economy with perfect foresight having both spot and one period forward markets for assets in each market period. In such a model any equilibrium sequence of spot and forward prices will have the property that expected future spot prices and current forward prices of the same maturity are the same. If not, arbitrarily large profits could be made with certainty. In addition any equilibrium sequence of consumption, production, employment, taxing and public spending plans can be realized using either just the sequence of spot markets or the sequence of one period forward markets.

The balance sheet constraints and "psychological" budget constraints for the business sector and the government sector are derived in the same way as for the household sector. They are given below together with the economy-wide constraints.

The business sector:

42. $E^s(t,t,t,h)-E(t)\equiv 0$

43. $p(t,t,h)Q^s(h,t)+p_E^*(t,t+h,t+h,h)[E^s(t,t+h,t+h,h)-E^s(t,t,t,h)]$
$\equiv w(t,t,h)N^d(h,t)+\Pi^f(h,t)+p(t,t,h)I^d(h,t)$

43'. $p(t,t,h)Q(h,t)+p_E(t+h,t+h,h)[E(t+h)-E(t)]\equiv w(t,t,h)N(h,t)+\Pi(h,t)+p(t,t,h)I(h,t)$

The government sector:

44. $M^s(t,t,t,h)-M(t)+B^s(t,t,t,h)-B(t)\equiv 0$

45. $M^s(t,t+h,t+h,h)-M^s(t,t,t,h)+B^s(t,t+h,t+h,h)-B^s(t,t,t,h)+T^g(h,t)\equiv$
$p(t,t,h)G^d(h,t)+r(t,t,h)B(t)h$

45'. $M(t+h)-M(t)+B(t+h)-B(t)+T(h,t)\equiv p(t,t,h)G(h,t)+r(t,t,h)B(t)h$

The whole economy:

46. $M^s(t,t,t,h)-M^d(t,t,t,h)+B^s(t,t,t,h)-B^d(t,t,t,h)+p_E(t,t,h)[E^s(t,t,t,h)-E^d(t,t,t,h)] \equiv 0$

47. $$M^s(t,t+h,t+h,h)-M^s(t,t,t,h)-[M^d(t,t+h,t+h,h)-M^d(t,t,t,h)]$$
$$+B^s(t,t+h,t+h,h)-B^s(t,t,t,h)-[B^d(t,t+h,t+h,h)-B^d(t,t,t,h)]$$
$$+p_E^*(t,t+h,t+h,h)\left(E^s(t,t+h,t+h,h)-E^s(t,t,t,h)-[E^d(t,t+h,t+h,h)-E^d(t,t,t,h)]\right)$$
$$+w(t,t,h)[N^s(h,t)-N^d(h,t)]+p(t,t,h)[Q^s(h,t)-C^d(h,t)-I^d(h,t)-G^d(h,t)]$$
$$\equiv \Pi^f(h,t)-\Pi^H(h,t)+T^H(h,t)-T^g(h,t)$$

The market equilibrium conditions in the beginning-of-period model are:

48a. $M^s(t,t,t,h) = M^d(t,t,t,h)$ $\quad [=M(t)]$

48b. $B^s(t,t,t,h) = B^d(t,t,t,h)$ $\quad [=B(t)]$

48c. $E^s(t,t,t,h) = E^d(t,t,t,h)$ $\quad [=E(t)]$

48d. $N^s(h,t) = N^d(h,t)$ $\quad [=N(h,t)]$

48e. $Q^s(h,t) = C^d(h,t)+I^d(h,t)+G^d(h,t)$ $\quad [Q^s=Q, C^d=C, I^d=I, G^d=G]$

While no Walras' Law of markets can be applied to the values of the excess demands in all markets in the usual interpretation of the beginning-of-period model, something fairly close to Walras' Law is implied by 47) and the assumption of homogeneous expectations about taxes and dividends. By defining planned household saving as in 49) we preserve the national income identity that $\tilde{S} \equiv Y-C^d$.

49. $$\tilde{S}(h,t)\equiv M^d(t,t+h,t+h,h)-M^d(t,t,t,h)+B^d(t,t+h,t+h,h)$$
$$-B^d(t,t,t,h)+p_E^*(t,t+h,t+h,h)[E^d(t,t+h,t+h,h)-E^d(t,t,t,h)]$$

This expression is the same as the end-of-period one in equation 30, except that the non-existent forward contracts and forward prices have been replaced by planned future spot contracts and expected future spot prices. Thus in the beginning-of-period model the saving decision is again known as soon as the asset demand functions and expectations about the future are given.

The continuous trading case.

When we take the limit as $h \to 0$ of 40) we obtain:

50. $M(t)+B(t)+p_E(t,t,0)E(t) \equiv M^d(t,t,t,0)+B^d(t,t,t,0)+p_E(t,t,0)E^d(t,t,t,0)$

Subtracting 50) from 40), dividing by h and taking the limit as $h \to 0$ yields

51. $M_4^d(t,t,t,0)+B_4^d(t,t,t,0)+p_E(t,t,0)E_4^d(t,t,t,0)+p_{E,3}(t,t,0)[E^d(t,t,t,0)-E(t)] \equiv 0$

The sum of the rates at which household demand for the three assets changes when the length of the interval between markets (i.e. the frequency with which markets open) changes will be zero only if the asset markets are in instantaneous equilibrium and households are in instantaneous portfolio equilibrium $[E^d(t,t,t,0)=E(t)]$. The reason is that a change in this market interval will not change the households' endowment $[M(t), B(t), E(t)]$ at a point in time. It will, however, by changing p_E alter the value of the endowment. This will have to be allowed for unless asset endowments and instantaneous asset demands coincide, in which case both would be equally affected by the change in p_E. No condition like 51) holds for the continuous time version of the end-of-period model. Putting 40) and 41) together we obtain in the limit:

52. $M_2^d(t,t,t,0)+M_3^d(t,t,t,0)+B_2^d(t,t,t,0)+B_3^d(t,t,t,0)+p_E(t,t,0)[E_2^d(t,t,t,0)+E_3^d(t,t,t,0)]$
$\equiv w(t,t,0)n^s(t)+\pi^H(t)+r(t,t,0)B(t)-\tau^H(t)-p(t,t,0)c^d(t)$

Balance sheet constraints and "psychological" budget constraints for the business sector, the government sector and the economy as a whole can be derived in an analogous manner. They are omitted here for reasons of space. The market equilibrium conditions in the continuous trading version of the beginning-of-period model, obtained by considering the limits of 48a) and 48e) when $h \to 0$ are the same as in the end-of-period model [equations 27a-27e]. Compare the continuous trading balance sheet constraints and flow budget constraints of the end-of-period and the beginning-of-period model. 19) and 50) are the same. 20) and 52) will be equivalent if markets clear i.e. $E^d(t,t,t,0)=E(t)$ and if $M_2^d(t,t,t,0)+B_2^d(t,t,t,0)+p_E(t,t,0)E_2^d(t,t,t,0) \equiv 0$.[5] The most plausible sufficient condition for the second condition to hold is $M_2^d=B_2^d=E_2^d=0$. Asset demands for delivery at some fixed future date do not vary when the market date is varied. This "irrelevance of the market date" requirement will be satisfied if transactions costs are not dependent on the market date and if there is perfect foresight. Together the two equivalence conditions can be referred to as the perfect foresight and perfect markets requirement. Again this analysis can be repeated for the other sectors and for the economy as a whole with similar results. The conclusion is the plausible one that the continuous trading versions of the beginning-of-period model and the end-of-period model are equivalent if markets clear and the market date is irrelevant, or more suggestively although rather less precise, the two are equivalent when there are perfect markets and perfect foresight. Under these condtions, the beginning-of-period continuous time saving rate $\tilde{s}(t) \equiv M_2^d(t,t,t,0)+M_3^d(t,t,t,0)+B_2^d(t,t,t,0)$ $+B_3^d(t,t,t,0)+p_E(t,t,0)\,[E_2^d(t,t,t,0)+E_3^d(t,t,t,0)]$ will also be equivalent to the end-of-period continuous time saving rate given by $s(t)$ in 38).

This equivalence result applies not only in the limit, as $h \to 0$, but also for finite h. The discrete time begining-of-period and end-of-period models will be equivalent if the following holds:

53a. $M^d(t,t,t+h,h)=M^d(t,t+h,t+h,h)=M(t+h)=M^s(t,t,t+h,h)=M^s(t,t+h,t+h,h)$.

53b. $B^d(t,t,t+h,h)=M^d(t,t+h,t+h,h)=B(t+h)=B^s(t,t,t+h,h)=B^s(t,t+h,t+h,h)$.

53c. $E^d(t,t,t+h,h)=E^d(t,t+h,t+h,h)=E(t+h)=E^s(t,t,t+h,h)=E^s(t,t+h,t+h,h)$.

It is instructive to express the first three items of 53c) in terms of real demands and supplies, i.e.

54. $$\frac{p^*(t,t+h,t+h,h)}{p_E^*(t,t+h,t+2h,h)}\, j(t,t,t+h,h)=\frac{p^*(t,t+h,t+h,h)}{p_E^*(t,t+2h,t+2h,h)}\, j(t,t+h,t+h,h)=E(t+h)$$

54) states that the equity market should clear, that real one-period forward equity demand equals real planned spot demand for equity one period from now and that the one period forward price of equity expected, at t, to prevail in the market at $t+h$ equals the spot price of equity expected, at t, to prevail in the market at $t+2h$. Perfect foresight and perfect markets are therefore sufficient for equivalence of the end-of-period and the beginning-of-period models in the discrete period model as in the continuous trading model.

5. Conclusion

This paper has been concerned with the way in which time enters into sequential temporary equilibrium models.[6] The presentation abstracted from real-world complexity in many drastic ways. A major abstraction was involved in assigning a market period of equal duration to all goods, services and financial claims. Still it was found that the minimal representation of the time structure of asset demand functions and price expectation functions required four dimensions: the date at which plans were made (expectations were formed); the market date to which plans (expectations) refer; the maturity or duration of the forward contracts concluded in the market (the "delivery period") and the duration of the interval between markets. When we considered the limiting properties as h went to zero of discrete time end-of-period and beginning-of-period models, both the maturity of the forward contracts and the length of the interval between markets went to zero. Further analysis is required of the case in which continuous spot and forward trading can occur with forward contracts of finite maturity. The natural representation of such an economic system is in terms of systems of mixed differential-difference equations or general integral equations. This may seem a daunting prospect, but it is also an exciting one, as it could lead to a fuller understanding of the essential properties of sequence economies.

This paper grew out of my Ph.D. Thesis (Buiter [1975]). Over the past four years I have had many useful exchanges on this subject with Duncan Foley, Martin Hellwig, Dwight Jaffee, Gary Smith, James Tobin and Geoffrey Woglom.

FOOTNOTES

[1] Δt_3 in 7) need not be the same as in 1). For our purposes we can ignore this.

[2] To derive 38) and 39) we use the continuous time balance sheet constraint. The term $2p^*_{E,3}$ in 39) comes from using the approximation:

$$p^*_E(t,t+h,t+2h,h) \simeq p^*_E(t,t+h,t+h,h)+p^*_{E,3}(t,t+h,t+h,h)h \ .$$

[3] See Tobin [1977].

[4] If income and expenditure flows between t and t+h were to constitute sources and uses of funds that could be allocated to beginning-of-period asset stocks at t, beginning-of-period and end-of-period models would be virtually identical. Beginning-of-period asset demands like $M^d(t,t,t,h)$ would be the asset stocks people plan to carry over to the next period, i.e. in equilibrium, $M^d(t,t,t,h) = M(t+h)$ etc. A single budget constraint would link beginning-of-period asset demands at t and other income and outlay flows from t to t+h. The only difference with the end-of-period model is that in the end-of-period model asset sellers carry the asset stocks from the market date at t to the delivery date at t+h. In the beginning-of-period model asset purchasers carry the asset stocks from one market date at t to the next market date at t+h.

[5] Note that 51) is used to establish this equivalence.

[6] "Equilibrium" should be interpreted broadly enough to include quantity constrained excess supply or demand situations.

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